# A $p$-adically entire function with integral values on $\mathbb{Q}_p$ and additive characters of perfectoid fields

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# Outline

# The function $\Psi_p$

A prime $p$ is fixed all over. We consider the formal solution

$$\Psi(T) = \Psi_p(T) = T + \sum_{i=2}^{\infty} a_i T^i \in \mathbb{Z}[[T]] \ ,$$

to the functional equation

$$(*) \qquad \sum_{j=0}^{\infty} p^{-j} \Psi(p^j T)^{p^j} = T \ .$$

The following facts were proven in my thesis (Padova 1974 - Ann. Sc. Norm. Sup. 1975)

1. $\Psi_p$ is $p$-adically entire;
2. $\Psi_p(\mathbb{Q}_p) \subset \mathbb{Z}_p$;
3. for any $i \in \mathbb{Z}$ and $x \in \mathbb{Q}_p$, if we define $x_{-i} := \Psi_p(p^i x)$ mod $p \in \mathbb{F}_p$ then

$$x = \sum_{i>>-\infty}^{\infty} [x_i]\, p^i \in \mathrm{W}(\mathbb{F}_p)[1/p] = \mathbb{Q}_p \ ,$$

where $[t]$, for $t \in \mathbb{F}_p$, is the Teichmüller representative of $t$ in $\mathrm{W}(\mathbb{F}_p) = \mathbb{Z}_p$.
4. $\Psi_p$ trivializes the addition law of Witt covectors with coefficients in the Fréchet algebra $\mathbb{Q}_p\{x, y\}$ of entire functions of $x$ and $y$.

$$(\ldots, \Psi(p^2x), \Psi(px), \Psi(x)) + (\ldots, \Psi(p^2y), \Psi(py), \Psi(y)) =$$

$$(\ldots, \Psi(p^2(x+y)), \Psi(p(x+y)), \Psi(x+y)) \ .$$

Watch out : This is a sum of Witt covectors ! We will explain later what this means.

Some of these results admit an elementary proof. For example

## Proposition

*The functional equation* $(*)$ *has a unique solution* $\Psi = \Psi_p \in T\mathbb{Z}[[T]]$ *.*

**Proof.**

We endow $T\mathbb{Z}[[T]]$ of the $T$-adic topology. It is clear that, for any $\varphi \in T\mathbb{Z}[[T]]$, the series $T - \sum_{j=1}^{\infty} p^{-j}\varphi(p^j T)^{p^j}$ converges in $T\mathbb{Z}[[T]]$ and that the map

$$\varphi \longmapsto T - \sum_{j=1}^{\infty} p^{-j}\varphi(p^j T)^{p^j} ,$$

is a contraction of the complete metric space $T\mathbb{Z}[[T]]$. So, this map has a unique fixed point which is $\Psi(T)$. □

It is also easy to prove that

**Proposition**

*The series $\Psi(T)$ is entire.*

Proof.

Since $\Psi \in T\mathbb{Z}[[T]]$, we deduce that $\Psi$ converges for $v_p(T) > 0$. On the other hand, it is clear that the coefficient of $T$ in $\Psi(T)$ is 1. Therefore, for $v_p(T) > 0$, $v_p(\Psi(T)) = v_p(T)$.

Suppose $\Psi$ converges for $v_p(T) > \rho$. Then, for $j \geq 1$, $\Psi(p^j T)^{p^j}$ converges for $v_p(T) > \rho - 1$. Moreover, if $j > -\rho + 1$ and $v_p(T) > \rho - 1$, we have

$$v_p(p^{-j}\Psi(p^j T)^{p^j}) \geq -j + p^j(j + \rho - 1) \ ,$$

and this last term $\to +\infty$, as $j \to +\infty$.

This shows that the series $T - \sum_{j=1}^{\infty} p^{-j}\Psi(p^j T)^{p^j}$ converges uniformly for $v_p(T) > \rho - 1$, so that its sum, which is $\Psi$, is analytic for $v_p(T) > \rho - 1$. It follows immediately from this that $\Psi$ is an entire function. □

# Outline

## Proposition

*For any $a \in \mathbb{Q}_p$, $\Psi_p(a) \in \mathbb{Z}_p$.*

Proof. Let $a \in \mathbb{Z}_p$. We define by induction the sequence $\{a_i\}_{i=0,1,\dots}$ :

$$a_0 = a \ , \ \ a_1 = p^{-1}(a_0 - a_0^p) \ , \ \ a_2 = p^{-2}(a_0^p - a_0^{p^2}) + p^{-1}(a_1 - a_1^p) \ ,$$

$$a_i = \sum_{j=0}^{i-1} p^{j-i}(a_j^{p^{i-j-1}} - a_j^{p^{i-j}}) \ .$$

Since, for any $a, b \in \mathbb{Z}_p$, if $a \equiv b \mod p$, then $a^{p^n} \equiv b^{p^n} \mod p^{n+1}$, while $a \equiv a^p \mod p$, we see that $a_i \in \mathbb{Z}_p$, for any $i$.

We then see by induction that, for any $i$,

$$a_i = p^{-i}(a - \sum_{j=0}^{i-1} p^j a_j^{p^{i-j}}) \text{ or, equivalently, } a = \sum_{j=0}^{i} p^j a_j^{p^{i-j}} .$$

More precisely, if we stick in the formula which defines $a_i$, namely

$$p^i a_i = \sum_{j=0}^{i-1} p^j a_j^{p^{i-j-1}} - \sum_{j=0}^{i-1} p^j a_j^{p^{i-j}}$$

the $(i-1)$-st step of the induction, namely, $a = \sum_{j=0}^{i-1} p^j a_j^{p^{i-j-1}}$, we get

$$p^i a_i = a - \sum_{j=0}^{i-1} p^j a_j^{p^{i-j}} ,$$

which is precisely the $i$-th inductive step.

From the functional equation we have

$$\Psi(p^{-i}a) \equiv p^{-i}a - \sum_{\ell=1}^{i} p^{-\ell}\Psi(p^{\ell}p^{-i}a)^{p^{\ell}} =$$
$$p^{-i}(a - \sum_{j=0}^{i-1} p^{j}\Psi(p^{-j}a)^{p^{i-j}}) \mod p\mathbb{Z}_p \ .$$

We then see by induction that

$$\Psi(p^{-i}a) \equiv a_i \mod p\mathbb{Z}_p \ ,$$

which proves the statement. In fact, assume $\Psi(p^{-j}a) \equiv a_j \mod p\mathbb{Z}_p$, for $j = 0, 1, \ldots, i-1$, and plug this information in the previous formula. We get

$$\Psi(p^{-i}a) \equiv p^{-i}a - \sum_{\ell=1}^{i} p^{-\ell} a_{i-\ell}^{p^{\ell}} = p^{-i}(a - \sum_{j=0}^{i-1} p^{j} a_j^{p^{i-j}}) = a_i \ ,$$

which is the $i$-th inductive step.

We now know more.

**Theorem**

*The valuation polygon of* $\Psi_p$

$$\mu \longmapsto v(f,\mu) = \inf_{i\in\mathbb{Z}} i\,\mu + v(a_i)$$

*goes through the origin, has slope 1 for* $\mu > -1$*, and slope* $p^j$*, for* $-j-1 < \mu < -j$*,* $j = 1, 2, \ldots$ *.*

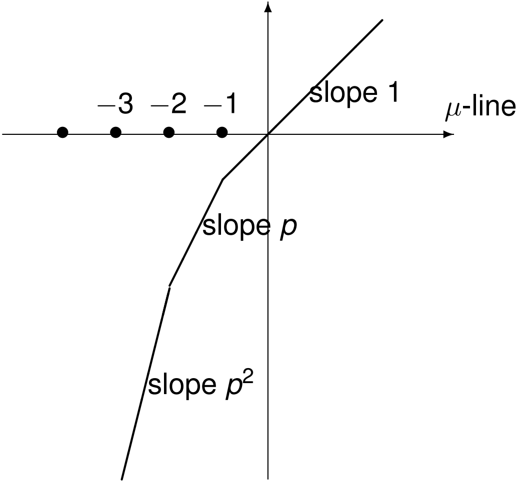


Figure : The valuation polygon of $\Psi_p$.

## Corollary

*The Newton polygon* $\mathrm{Nw}(\Psi)$ *has vertices at the points*

$$V_i := (-p^i, i\,p^i - \frac{p^i - 1}{p - 1}) = (-p^i, i\,p^i - p^{i-1} - \cdots - p - 1)\ .$$

*The equation of the side joining the vertices* $V_i$ *and* $V_{i-1}$ *is*

$$Y = -iX - \frac{p^i - 1}{p - 1}\ ;$$

*its projection on the* $X$*-axis is the segment* $[-p^i, -p^{i-1}]$*. So,* $\mathrm{Nw}(\Psi)$ *has the form described in the next figure.*

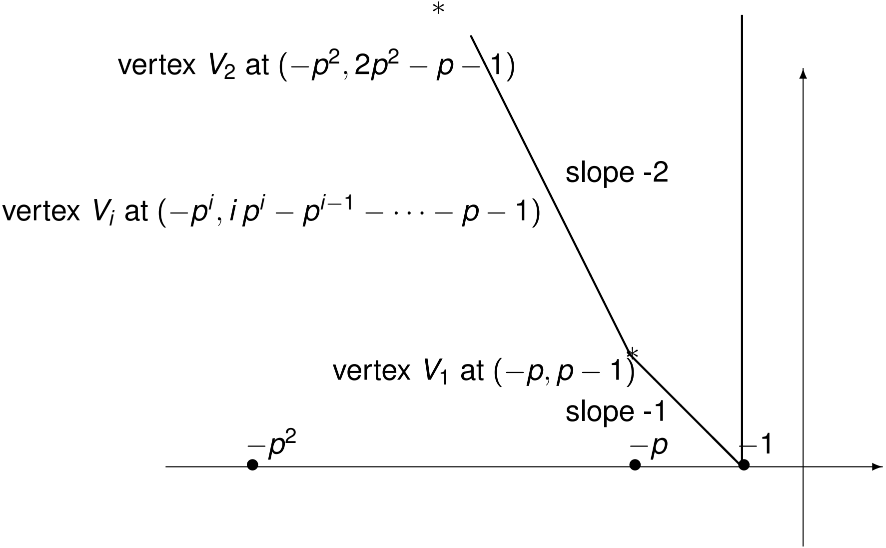


Figure : The Newton polygon $\mathrm{Nw}(\Psi_p)$ of $\Psi_p$.

## Corollary

*For any* $i = 0, 1, \ldots$, *the map* $\Psi = \Psi_p$ *induces finite coverings of degree* $p^i$,

$$\Psi : \{\, x \in \mathbb{C}_p \,|\, v_p(x) > -i-1 \,\} \longrightarrow \{\, x \in \mathbb{C}_p \,|\, v_p(x) > -\frac{p^{i+1}-1}{p-1} \,\} \,,$$

*(in particular, an isomorphism*

$$\Psi : \{\, x \in \mathbb{C}_p \,|\, v_p(x) > -1 \,\} \xrightarrow{\sim} \{\, x \in \mathbb{C}_p \,|\, v_p(x) > -1 \,\} \,,$$

*for* $i = 0$*).*

More precisely, $\Psi$ induces finite maps of degree $p^i$

$$\Psi :\{ x \in \mathbb{C}_p \mid -(i+1) < v_p(x) < -i \} \longrightarrow \{ x \in \mathbb{C}_p \mid -\frac{p^{i+1}-1}{p-1} < v_p(x) < -\frac{p^i-1}{p-1} \} ,$$

and finite maps of degree $p^{i+1} - p^i$

$$\Psi : \{ x \in \mathbb{C}_p \mid v_p(x) = -i-1 \} \longrightarrow \{ x \in \mathbb{C}_p \mid -\frac{p^{i+1}-1}{p-1} \leq v_p(x) \} .$$

*The function*

$$\Psi_p : \mathbb{A}^1_{\mathbb{Q}_p} \to \mathbb{A}^1_{\mathbb{Q}_p}$$

*is a quasi-finite covering of the Berkovich affine line over $\mathbb{Q}_p$ by itself.* Aside from ramification, its behaviour is very similar to the one of the map $\log : D_{\mathbb{Q}_p}(1, 1^-) \to \mathbb{A}^1_{\mathbb{Q}_p}$, where $D_{\mathbb{Q}_p}(1, 1^-)$ is the open unit disk in $\mathbb{A}^1_{\mathbb{Q}_p}$. *I believe, but cannot prove, that, after base change to $\mathbb{C}_p$, $\Psi_p$ is a (ramified) Galois abelian covering.*

# Outline

# Perfectoid fields

We recall that a *perfectoid field* is a non-discretely valued non-archimedean field $K$ such that the Frobenius map of $K^\circ/pK^\circ$ is surjective. For any perfectoid field $(K, |\,|)$, one defines the *tilt* $K^\flat = \varprojlim(K, x \mapsto x^p)$ of $K$. It is a perfect non-archimedean extension of $\widetilde{K}$, $K^\flat = \widetilde{K}\widehat{((t^{1/p^\infty}))}$. The $t$-adic valuation of the element $\varpi = (\varpi^{(0)}, \varpi^{(1)}, \dots) \in K^\flat$, with $\varpi^{(i)} \in K$, $(\varpi^{(i+1)})^p = \varpi^{(i)}$, is, by definition, $v_K(\varpi^{(0)})$.
If $K$ is of characteristic $p$, then $K^\flat = K$.

A *pseudo-uniformizer* $\varpi = (\varpi^{(0)} \leftarrow \varpi^{(1)} \leftarrow \dots)$ of $K^\flat$ is an element of $(K^\flat)^{\circ\circ}$. For any $i = 0, 1, 2, \dots$, we define $\varpi_i = (\varpi^{(i)} \leftarrow \varpi^{(i+1)} \leftarrow \dots)$, so that $\varpi_i$ is the unique $p^i$-th root of $\varpi$ in $K^\flat$. We consider the element

$$\pi = \pi(\varpi) := \sum_{i \geq 0} \varpi^{(i)} p^i + \sum_{i<0} (\varpi^{(0)})^{p^{-i}} p^i \in K \ .$$

Notice that this is a convergent sum in $K$ and that

$$\pi(\varpi^{p^i}) = p^i \, \pi(\varpi) \ \ , \ \ \pi(\varpi_i) = p^{-i} \, \pi(\varpi) \ .$$

We will use the formula of Dieudonné

$$\prod_{i=0}^{\infty} F(x_i T^{p^i}) = \exp \sum_{i=0}^{\infty} x^{(i)} T^{p^i} = 1 + \sum_{i=1}^{\infty} g_i(x_0, x_1, \ldots, x_{[\log_p i]}) T^i \ ,$$

where $F(T) = \exp(\sum_{i=0}^{\infty} T^{p^i}/p^i) \in \mathbb{Z}_{(p)}[[T]]$ is the Artin-Hasse exponential series and $x^{(i)} = \sum_{n=0}^{i} p^{n-i} x_n^{p^{n-i}}$ is the ghost component of the Witt vector $(x_0, x_1, \ldots)$. The plan is to introduce variables $x_i, y_i$ with negative indices $i$ and to prolong that formula into (for $S = \mathbb{Z}[1/p] \cap \mathbb{R}_{\geq 0}$)

$$\prod_{i=-\infty}^{\infty} F(x_i T^{p^i}) = \exp \sum_{i=-\infty}^{\infty} x^{(i)} T^{p^i} =$$

$$1 + \sum_{q \in S} g_q(\ldots, x_{[\log q]-1}, x_{[\log q]}) T^q \ .$$

Then, we want to specialize $x_i \mapsto \Psi_p(p^{-i}x)$, for any $i \in \mathbb{Z}$, and $T^{1/p^i} \mapsto \varpi^{(i)}$, for any $i = 0, 1, 2, \dots$ and to use the integrality properties of $\Psi_p$ (i.e. $\Psi_p(\mathbb{Q}_p) \subset \mathbb{Z}_p$), to show that the map $x \mapsto \exp \pi(\varpi)x$, a priori only defined for

$$v_p(x) > \frac{1}{p-1} - v_p(\pi(\varpi)) \ ,$$

canonically extends to a continuous additive character

$$\Psi_\varpi : \mathbb{Q}_p \to 1 + K^{\circ\circ} \ .$$

Note that such a character is an element of the inverse limit of $1 + K^{\circ\circ}$ under the $p$-th power map, which is the same as $1 + (K^\flat)^{\circ\circ}$. We then obtain a map $\varpi \mapsto \Psi_\varpi$ from the open unit disk at 0 to the open disk at 1, both over $K^\flat$.

On the other hand, for any $i = 0, 1, 2, \dots,$

$$\Psi_{\varpi^{p^i}}(x) = \Psi_{\varpi}(x)^{p^i} \ ,$$

as convergent series in a neighborhood of $x = 0$, and for any fixed $x \in \mathbb{Q}_p$. So, the previous map

$$\begin{aligned} (K^\flat)^{\circ\circ} &\to 1 + (K^\flat)^{\circ\circ} \\ \varpi &\mapsto \Psi_\varpi \end{aligned}$$

commutes with Frobenius. Following a suggestion of Jared Weinstein, I prove that this map is induced by the Artin-Hasse function in characteristic $p$. In particular, it is $\mathbb{F}_p$-analytic, is independent of $K$, and it is an isomorphism.

# Outline

The addition of Witt vectors is given by

$$(x_0, x_1, \ldots, x_m) + (y_0, y_1, \ldots, y_m) =$$
$$(\varphi_0(x_0; y_0), \varphi_1(x_0, x_1; y_0, y_1), \ldots, \varphi_m(x_0, x_1, \ldots, x_m; y_0, y_1, \ldots, y_m)) \,,$$

where

$$\varphi_i(x_0, x_1, \ldots, x_i; y_0, y_1, \ldots, y_i) \in \mathbb{Z}[x_0, x_1, \ldots, x_i, y_0, y_1, \ldots, y_i] \,.$$

This is done in such a way that, if

$$x^{(i)} = x_i + p^{-1} x_{i-1}^{p} + \cdots + p^{-i} x_0^{p^i}$$

is the *i-th ghost component* of the vector $x$, then

$$(x + y)^{(i)} = x^{(i)} + y^{(i)} \,.$$

This makes a smooth affine group (ring in fact) $W_m$ over $\mathbb{Z}$ and have $W = \varprojlim W_m$, as group (ring, in fact) functors. Barsotti introduced the ring functor of *unipotent bivectors*

$$\mathrm{BW}^u = \varinjlim(\mathrm{W} \xrightarrow{V} \mathrm{W} \xrightarrow{V} \dots)$$

where $V(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$. It is most convenient to write the elements of $\mathrm{BW}(R)$, for any ring $R$, as *Witt bivectors* with components in $R$

$$(\dots, 0, 0, x_{-n}, \dots, x_{-1}; x_0, x_1, \dots) \, .$$

In particular, $\mathrm{BW}^u(\mathbb{F}_p) = \mathbb{Q}_p$.

Let us attribute the weight $p^i$ to the variables $x_i$ and $y_i$. Then the polynomial $\varphi_m(x_0, x_1, \ldots, x_m; y_0, y_1, \ldots, y_m)$ is isobaric of weight $p^m$. Moreover, for any $i \geq 1$

$$\varphi_i(x_0, x_1, \ldots, x_i; y_0, y_1, \ldots, y_i) - \varphi_{i-1}(x_1, \ldots, x_i; y_1, \ldots, y_i)$$

is divisible by $x_0 y_0$. The addition operation of unipotent Witt bivectors is then determined by a single expression

$$\Phi(\ldots, x_{-n-1}, x_{-n}, \ldots, x_{-1}, x_0; \ldots, y_{-n-1}, y_{-n}, \ldots, y_{-1}, y_0) = \lim_{m \to +\infty} \varphi_m(x_{-m}, x_{1-m}, \ldots, x_{-1}, x_0; y_{-m}, y_{1-m}, \ldots, y_{-1}, y_0) \,,$$

which in fact is eventually constant since $x$ and $y$ are eventually 0, due to the previous congruences.

The addition of two unipotent bivectors is given by

$$(\dots, x_{-1}; x_0, x_1, \dots) + (\dots, y_{-1}; y_0, y_1, \dots) =$$
$$(\dots, \Phi(\dots, x_{-3}, x_{-2}; \dots, y_{-3}, y_{-2}), \Phi(\dots, x_{-2}, x_{-1}; \dots, y_{-2}, y_{-1});$$
$$\Phi(\dots, x_{-1}, x_0; \dots, y_{-1}, y_0), \Phi(\dots, x_0, x_1; \dots, y_0, y_1), \dots)$$

We still have the formula

$$(x + y)^{(i)} = x^{(i)} + y^{(i)} \, .$$

where

$$x^{(i)} = x_i + p^{-1} x_{i-1}^{p} + \dots + p^{-i} x_0^{p^i} + \dots$$

involving the *i-th ghost component* of the unipotent bivector $x$.

# Outline

We recall that the *hyperexponential group* is the affine group $\mathrm{H}$ (over $\mathbb{Z}$), such that, for any ring $R$,

$$\mathrm{H}(R) = \{(a_1, a_2, \dots) \,|\, a_i \in R\} \ ,$$

viewed as an abelian group for the addition law $(a_1, a_2, \dots) + (b_1, b_2, \dots) = (c_1, c_2, \dots)$ if

$$(1 + \sum_{i=1}^{\infty} a_i T^i)\,(1 + \sum_{i=1}^{\infty} b_i T^i) = 1 + \sum_{i=1}^{\infty} c_i T^i \ ,$$

for the usual product in the formal power series ring $R[[T]]$. We consider a subgroup $\mathrm{C}$ of $\mathrm{H}$ restricted to $\mathbb{Z}_{(p)}$-rings, called the *Cartier group*, namely the subgroup generated by elements of the form $1 + \sum_{i=0}^{\infty} a_{p^i} T^{p^i}$, in which case the only relevant entries of $(a_1, a_2, \dots) \in \mathrm{C}(R)$ are $(a_1, a_p, a_{p^2}, \dots)$. It will however be more convenient for us to use all the entries of $(a_1, a_2, \dots)$.

We now exploit the isomorphism between the two affine groups $\mathrm{W}$ and $\mathrm{C}$ restricted to $\mathbb{Z}_{(p)}$. This is essentially deduced from the formula of Dieudonné

$$\prod_{i=0}^{\infty} F(x_i T^{p^i}) = \exp \sum_{i=0}^{\infty} x^{(i)} T^{p^i} = 1 + \sum_{i=1}^{\infty} g_i(x_0, x_1, \ldots, x_{[\log_p i]}) T^i \ ,$$

where $F(T) = \exp(\sum_{i=0}^{\infty} T^{p^i}/p^i) \in \mathbb{Z}_{(p)}[[T]]$ is the Artin-Hasse exponential series and $x^{(i)} = \sum_{n=0}^{i} p^{n-i} x_n^{p^{n-i}}$ is the ghost component of the Witt vector $(x_0, x_1, \ldots)$ (remember that it must be *divided by* $p^i$ w.r.t. the common use !). The identity, which is proved by computations in $\mathbb{Q}[\underline{x}][[T]]$, takes however place in the ring $\mathbb{Z}[\underline{x}][[T]]$.

## Proposition

*The polynomials* $g_i(x_0, x_1, \ldots, x_{[\log_p i]}) \in \mathbb{Z}_{(p)}[x_0, x_1, \ldots, x_{[\log_p i]}]$ *provide an isomorphism of affine groups* $\mathrm{W}(R) \xrightarrow{\sim} \mathrm{C}(R)$*,*

$$(x_0, x_1, \ldots) \longmapsto (y_1, y_2, \ldots, y_i, \ldots) = (g_1(x_0), \ldots, g_i(x_0, x_1, \ldots, x_{[\log_p i]}), \ldots) .$$

## Lemma

*Let us attribute to the variable* $x_i$ *the weight* $p^i$*. Then*

1. *The polynomial* $g_i(x_0, x_1, \ldots, x_{[\log i]}) \in \mathbb{Z}_{(p)}[x_0, x_1, \ldots, x_{[\log i]}]$ *is isobaric of weight* $i$*;*
2. $g_i(x_0, x_1, \ldots, x_{[\log i]})$ *belongs to the ideal* $(x_0, x_1, \ldots, x_{v_p(i)})$*;*
3. *Suppose* $v_p(i) \geq n$*. Then*

$$g_i(x_0, x_1, \ldots, x_{[\log i]}) \equiv g_{ip^{-n}}(x_n, \ldots, x_{[\log i]}) \mod (x_0, \ldots, x_{n-1}) .$$

# Outline

We want to extend the definition of *bivector with components in R* to certain topological rings $R$. We define the index set

$$S := \{\, q = \frac{n}{p^j} \mid n, j \in \mathbb{Z}_{\geq 0} \,\} \subset \mathbb{Q}_{\geq 0} \quad , \quad S = S' \,\dot{\cup}\, \{0\} \; .$$

We consider indeterminates $\underline{x} = (\ldots, x_{-1}; x_0, x_1, \ldots)$ over $\mathbb{Q}$, and the rings

$$\mathcal{P} := \mathbb{Z}_{(p)}[\underline{x}] \subset \mathcal{P}_{\mathbb{Q}} := \mathbb{Q}[\underline{x}] \; .$$

We attribute the *weight* $p^i$ to the indeterminate $x_i$, and consequently a weight $s \in S$ to any monomial in the $\underline{x}$.

Let $\gamma$ denote the canonical PD-structure on the ideal $(p) \subset \mathbb{Z}_{(p)}$.

## Definition

*For any linearly topologized* $\mathbb{Z}_{(p)}$*-ring* $R$ *(resp. PD-algebra* $(R, \mathcal{J}, [\,])$ *over* $(\mathbb{Z}_{(p)}, (p), \gamma)$*), a sequence*

$$a = (\dots, a_{-2}, a_{-1}; a_0, a_1, \dots)$$

*of elements of* $R$ *(resp. of* $\mathcal{J}$*) is admissible (resp. PD-admissible) if, for any neighborhood* $U$ *of* $0$ *in* $R$ *and for any* $s \in S$*, there exists an* $m \in \mathbb{Z}$ *such that, for any monomial* $x_{i_1}^{e_1} \dots x_{i_r}^{e_r}$ *of weight* $s$ *and of positive degree in some* $x_i$*, with* $i \leq m$*,* $a_{i_1}^{e_1} \dots a_{i_r}^{e_r} \in U$ *(resp.* $a_{i_1}^{[e_1]} \dots a_{i_r}^{[e_r]} \in U$*).*

For simplicity, we ignore the *PD*-case. Similary, we define a family of *simultaneously admissible* sequences. The condition of simultaneous admissibility for two sequences

$$a = (\dots, a_{-2}, a_{-1}; a_0, a_1, \dots) \text{ and } b = (\dots, b_{-2}, b_{-1}; b_0, b_1, \dots)$$

in a complete linearly topologized $\mathbb{Z}_{(p)}$-ring $R$, is precisely what guarantees that the previous expression for the $i$-th component in the sum $a + b = c$ of *Witt bivectors with components in R*

$$c_i = \Phi(\dots, a_{i-1}, a_i; \dots, b_{i-1}, b_i) = \lim_{m \to +\infty} \varphi_m(a_{i-m}, \dots, a_{i-1}, a_i; b_{i-m}, \dots, b_{i-1}, b_i) ,$$

converges in $R$, for any $i$.

## Definition

*For any $s \in S$ and $m \in \mathbb{Z}_{\geq 0}$, let $\mathcal{P}_s$ (resp. $\mathcal{I}_{s,m}$) be the $\mathbb{Z}_{(p)}$-submodule of $\mathcal{P}$ generated by all monomials in the $\underline{x}$ of weight $s$ (resp. and of positive degree in some $x_i$, with $i \leq -m$).*

So, if $m \leq n$, $\mathcal{I}_{s,n} \subset \mathcal{I}_{s,m}$. We write $\mathcal{I}_s$ for $\mathcal{I}_{s,-v_p(s)}$. We have

$$\mathcal{P}_s \mathcal{I}_{t,m} \subset \mathcal{I}_{s+t,m} \text{ and } \mathcal{I}_s \mathcal{I}_t \subset \mathcal{I}_{s+t} \ ,$$

for $s, t \in S$, $m, n \in \mathbb{Z}_{\geq 0}$. So, the family of submodules $\mathcal{I}_{s,m}$ for $m \in \mathbb{Z}_{\geq 0}$ defines a fundamental system of neighborhoods of 0 for a $\mathbb{Z}_{(p)}$-linear topology of $\mathcal{P}_s$. We denote completions by $\widehat{\mathcal{P}}_s$, $\widehat{\mathcal{I}}_s$. We construct the *Rees ring*

$$\widehat{\mathcal{I}}^{\bullet} := \bigoplus_{q \in S} \widehat{\mathcal{I}}_q \ ,$$

For $s \in S'$ and $m \in \mathbb{Z}$, the subset

$$U_{s,m} = \{(a_q)_{q \in S} \in \bigoplus_{q \in S} \widehat{\mathcal{I}}_q \mid a_q \in \widehat{\mathcal{I}}_{q,m} \, \forall \, q < s \} \ ,$$

is an ideal of $\widehat{\mathcal{I}}^\bullet$. The family $\{U_{s,m}\}_{s,m}$ is a basis of open ideals in a linear topology of $\widehat{\mathcal{I}}^\bullet$ which coincides with the product topology. The completion of $\widehat{\mathcal{I}}^\bullet$ in this topology will be denoted by

$$\widehat{\mathcal{I}} = \prod_{s \in S} \widehat{\mathcal{I}}_s \ .$$

We regard $\widehat{\mathcal{I}}$, via the identification $(a_q)_q \longleftrightarrow \sum_{q \in S} a_q T^q$, as a ring $\sum_{q \in S} \widehat{\mathcal{I}}_q T^q$ of *S-power series* $\sum_{q \in S} a_q T^q$, with $a_q \in \widehat{\mathcal{I}}_q$ for any $q \in S$.

The point of all this is that the sequence $i \mapsto x_i$ is admissible in $\widehat{\mathcal{I}}$ and the sequences $i \mapsto x_i \widehat{\otimes} 1$ and $1 \widehat{\otimes} x_i$, are simultaneously admissible in $\widehat{\mathcal{I}} \widehat{\otimes} \widehat{\mathcal{I}}$. We make $\widehat{\mathcal{I}}$ into a complete topological Hopf algebra over $\mathbb{Z}_{(p)}$ by defining

$$\begin{aligned} \Delta_{\mathrm{BW}} : \widehat{\mathcal{I}} &\longrightarrow \widehat{\mathcal{I}} \,\widehat{\otimes}_{\mathbb{Z}_{(p)}} \widehat{\mathcal{I}} \\ x &\longmapsto x \widehat{\otimes} 1 + 1 \widehat{\otimes} x \ , \end{aligned}$$

where $x = (\dots, x_{-2}, x_{-1}; x_0, x_1, \dots)$, meaning that $x_i \mapsto (x \widehat{\otimes} 1 + 1 \widehat{\otimes} x)_i$, for any $i \in \mathbb{Z}$. We define the *group of Witt bivectors* as the group functor

$$\begin{aligned} \mathrm{BW} : \mathcal{AR}_{\mathbb{Z}_{(p)}} &\longrightarrow \mathcal{A}b \\ R &\longmapsto Hom_{\mathcal{AR}_{\mathbb{Z}_{(p)}}}(\widehat{\mathcal{I}}, R) \ . \end{aligned}$$

where $\mathcal{AR}_{\mathbb{Z}_{(p)}}$ is a certain category of complete $\mathbb{Z}_{(p)}$-linearly topologized rings.

## Proposition

*For any $q \in S$, the sequence $n \mapsto g_{p^n q}(x_{-n}, \ldots, x_{[\log q]-1}, x_{[\log q]})$ converges in $\widehat{\mathcal{I}}_q$. We set*

$$g_q(\ldots, x_{-1}; x_0, x_1, \ldots) := g_q(\ldots, x_{[\log q]-1}, x_{[\log q]}) :=$$
$$\lim_{n\to\infty} g_{p^n q}(x_{-n}, \ldots, x_{[\log q]-1}, x_{[\log q]}) \in \widehat{\mathcal{I}}_q \ .$$

Our first main result is

## Theorem

*The equality*

$$\prod_{i=-\infty}^{\infty} F(x_i T^{p^i}) = \exp \sum_{i=-\infty}^{\infty} x^{(i)} T^{p^i} =$$
$$1 + \sum_{q\in S} g_q(\ldots, x_{[\log q]-1}, x_{[\log q]}) T^q \ .$$

*holds in $\widehat{\mathcal{I}} = \sum_{q\in S} \widehat{\mathcal{I}}_q T^q$.*

### Lemma

*For any complete linearly topologized $\mathbb{Z}_p$-ring, the set $\mathrm{BC}(R)$ of $S$-power series $1 + \sum_{q \in S} a_q T^q$, with $a_q \in R$ which satisfy the following condition*

($\Phi$) *For any neighborhood $U$ of $0$ in $R$ and any $t \in S'$, there exists $\varepsilon > 0$ such that $a_q \in U$ for any $q \in S \cap (t - \varepsilon, t)$.*

*is naturally a group.*

Proof.
We define a multiplication in $\mathrm{BC}(R)$, as follows. Let $1+\sum_{q\in S} a_q T^q$ and $1+\sum_{q\in S} b_q T^q$ be in $\mathrm{BC}(R)$. It is easy to check that, for any $q\in S$, and any neighborhood $U$ of 0 in $R$, only a finite set of $q_1, q_2 \in S$, with $q_1+q_2=q$ are such that $a_{q_1} b_{q_2} \notin U$. We may then set $c_q = \sum_{q_1+q_2=q} a_{q_1} b_{q_2}$, a converging sum in $\mathcal{I}_q$. We then set

$$1+\sum_{q\in S} c_q T^q = (1+\sum_{q\in S} a_q T^q)(1+\sum_{q\in S} b_q T^q)\ .$$

It is clear that $\mathrm{BC}(R)$ with the latter multiplication, is a group. □

## Definition

$\mathrm{BC}(R)$ *is called the group of hyperexponential bivectors with coefficients in R.*

Of course, $R \mapsto \mathrm{BC}(R)$ is a group functor on complete linearly topologized $\mathbb{Z}_p$-rings.
Notice that, for any complete linearly topologized $\mathbb{Z}_p$-ring, and any bivector $(\dots, a_{-1}; a_0, a_1, \dots) \in \mathrm{BW}(R)$, the $S$-series

$$1 + \sum_{q \in S} g_q(\dots, a_{-1}; a_0, a_1, \dots) T^q \ ,$$

satisfies condition $(\Phi)$, hence is a hyperexponential bivector with coefficients in $R$. We have thus defined a morphism of group functors on complete linearly topologized $\mathbb{Z}_p$-rings,

$$\mathrm{Hex} : \mathrm{BW} \longrightarrow \mathrm{BC} \ ,$$

which we call the *Dieudonné map*.

Let $K$ be any closed subfield of $\mathbb{C}_p$; we denote by $w_r$, for any $r \in \mathbb{Z}$, the valuation of $K\{x\}$ given by

$$w_r(f) = \inf_{x \in p^{-r}\mathbb{C}_p^\circ} v(f(x)) \ .$$

## Definition

*Let $c, N$ be positive constants. We denote by $K\{x, T^S\}_{c,N}$ the set of $S$-power series $\sum_{q \in S} a_q T^q$, with $a_q \in K\{x\}$, such that*

1. $a_{pq}(px) = a_q(x)$*, for any* $q \in S$*;*
2. *for any* $r, v \in \mathbb{Z}$ *and* $C \in \mathbb{R}$*,*

$$w_r(a_q) \geq C - c(\max(qp^r, 1)^N - 1) \ ,$$

*for almost all* $q$ *with* $v(q) \leq v$*.*

In particular, $a_0 \in K$ and $a_q(0) = 0$, if $q \in S'$.

For any $\sum_{q\in S} a_q T^q$ and $\sum_{q\in S} b_q T^q$ in $K\{x, T^S\}_{c,N}$, the sum

$$c_q := \sum_{q_1+q_2=q} a_{q_1} b_{q_2}$$

converges in $K\{x\}$, along the filter of cofinite subsets of $S$, and the $S$-power series $\sum_{q\in S} c_q T^q$ is an element of $K\{x, T^S\}_{c,N}$. With the natural operations, $K\{x, T^S\}_{c,N}$ is a ring *of restricted $S$-power series of type* $(c, N)$ *with coefficients in* $K$. For any given $m \in \mathbb{Z}$ and $\sum_{q\in S} a_q T^q \in K\{x, T^S\}_{c,N}$, we define

$$|| \sum_{q\in S} a_q T^q ||_m := p^{-\gamma} \ ,$$

where

$$\gamma = \inf \Big( w_r(a_q) + c(\max(qp^r, 1)^N - 1) \,|\, q \in S \, , \; v(q) \leq m - r \Big) \ .$$

Then $|| \; ||_m$ is a norm on $K\{x, T^S\}_{c,N}$ compatible with the $p$-adic valuation of $K$. The family of norms $\{|| \; ||_m\}_{m\in\mathbb{Z}}$ makes $K\{x, T^S\}_{c,N}$ into a Fréchet algebra over the valued field $(K, | \; |_p)$.

A strong effort is required to prove that

### Theorem

*The specialization $x_i \mapsto \Psi_p(p^{-i}x)$, for any $i \in \mathbb{Z}$, defines a continuous morphism of topological rings*

$$\widehat{\mathcal{I}} \longrightarrow K\{x, T^S\}_{\frac{p}{p-1},1} \ .$$

*More precisely, we have*

$$w_r(G_q) \geq -v(q) + \max(\log_p q, -r) - p\frac{\max(p^r q, 1) - 1}{p-1} \ .$$

The functions $g_q(\dots, x_{[\log q]-1}, x_{[\log q]})$ now specialize, via $x_i \mapsto \Psi_p(p^{-i}x)$, *i.e.* $x^{(i)} \mapsto p^{-i}x$, to entire functions $G_q(x)$, with coefficients in $\mathbb{Z}_{(p)}$, satisfying $G_{qp}(px) = G_q(x)$, for any $q \in S$, such that $G_q(a) \in \mathbb{Z}_p$, for any $a \in \mathbb{Q}_p$. So, the previous equality becomes the following equality in $\mathbb{Q}_p\{x, T^S\}_{\frac{p}{p-1},1}$

$$\prod_{i=-\infty}^{\infty} F(\Psi(p^{-i}x)T^{p^i}) = \exp\left( (\sum_{i=-\infty}^{\infty} p^{-i}T^{p^i})x \right) = 1 + \sum_{q \in S} G_q(x)T^q \ .$$

We now pick a perfectoid field $K$, $\mathbb{Q}_p \subset K \subset \mathbb{C}_p$ and the pseudouniformizer $\varpi = (\varpi^{(i)})_{i=0,1,\dots}$ of $K^\flat$. For any $q \in S$, we set

$$(\pi^{(0)})^q := \lim_{j\to\infty} (\pi^{(j)})^{qp^j} \ .$$

Notice that $(\pi^{(0)})^q \to 0$ as $v_p(q) \to +\infty$. Then, we may specialize $T^{1/p^n} \mapsto \varpi^{(n)}$ and we conclude

$$\prod_{i=0}^{\infty} F(\Psi(p^{-i}x)(\pi^{(0)})^{p^i}) \prod_{i=1}^{\infty} F(\Psi(p^i x)\pi^{(i)}) =$$

$$\exp \pi(\varpi)x = 1 + \sum_{q\in S} (\pi^{(0)})^q G_q(x) \ .$$

as germs of $K$-analytic functions in a neighborhood of 0.

The last term of the previous equality restricted to $x \in \mathbb{Q}_p$ represents a convergent sum of uniformly continuous functions $\mathbb{Q}_p \to 1 + K^{\circ\circ}$ because $G_q(a) \in \mathbb{Z}_p$, for any $a \in \mathbb{Q}_p$. It is a homomorphism, since it is obtained from a homomorphism $\mathrm{W} \xrightarrow{\sim} \mathrm{C}$. So, this gives a formula for

$$\Psi_\varpi : (\mathbb{Q}_p, +) \to (1 + K^{\circ\circ}, \cdot) \ ,$$

that is for $\Psi_\varpi \in 1 + (K^\flat)^{\circ\circ}$.

In the end, we have given a canonical $p$-adic analytic construction of a canonical coherent choice of $\Psi_\varpi(p^{-n})$ among the $p^n$-th roots of $\exp(p^n \pi(\varpi))$, for $n >> 0$, and we set $\Psi_\varpi = (\Psi_\varpi(p^{-n}))_{n=0,1,\ldots}$. It is a simple matter to see that this map coincides with the Artin-Hasse exponential in characteristic $p$. It is therefore an isomorphism.