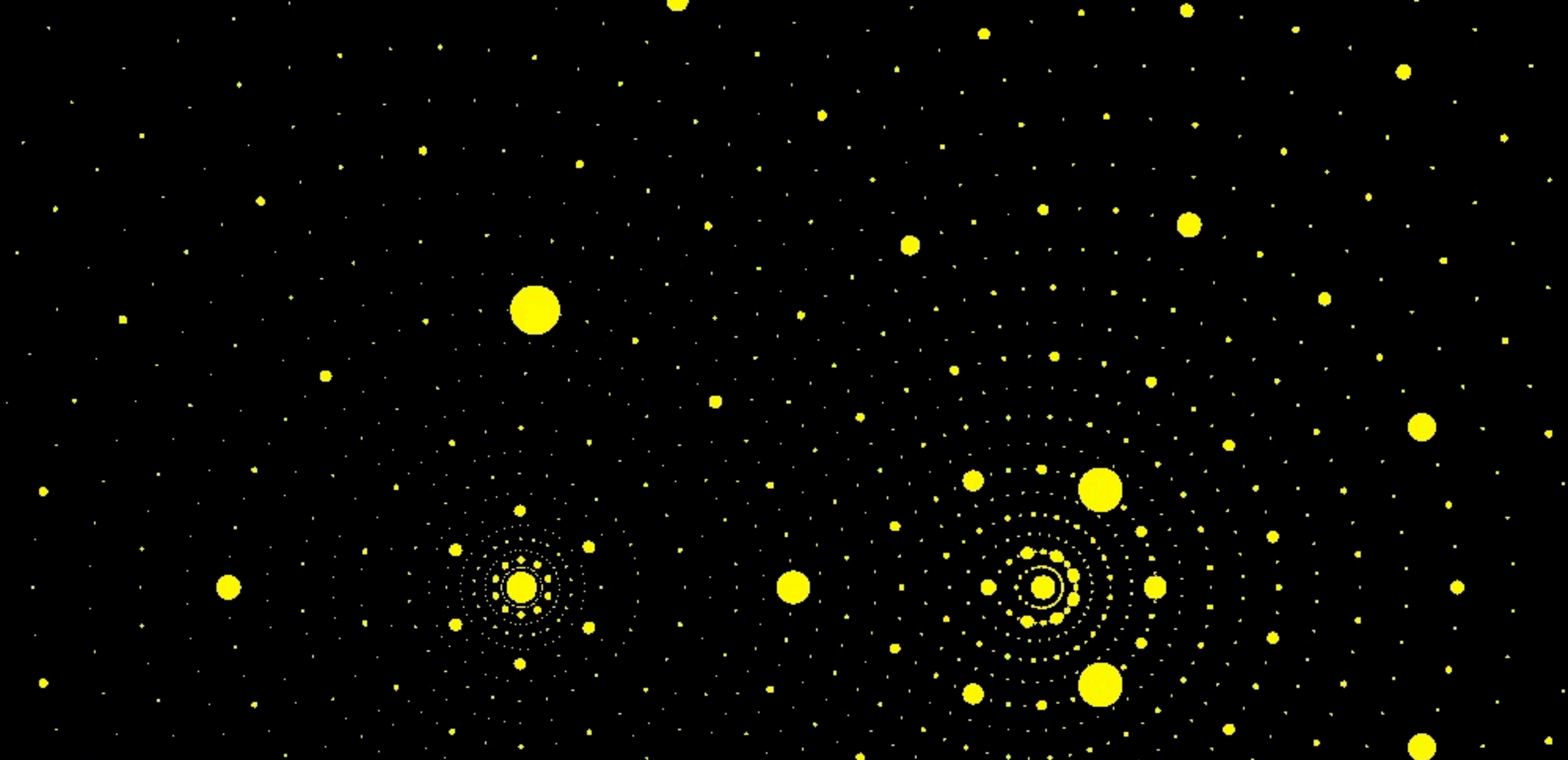
Variation of canonical height, illustrated

Laura DeMarco
Northwestern University



Theorem I.0.3. (Silverman, VCH I, 1992)

$P = (0,0)$

$E = \{y^2 + Txy + Ty = x^3 + 2Tx^3\}$

$$\hat{h}_{E_t}(P_t) = \frac{1}{15}\log t + \frac{2}{25}\log 2 + \frac{2}{25}\frac{(\log 2)^2}{\log(t^5/2)} + O(t^{-1}) \text{ for } t \in \mathbb{Z},\ t \to \infty$$

Variation of canonical height, illustrated

- Brief overview:
  families of elliptic curves
- Connections with dynamics
- pictures
- Rationality of canonical heights?
  (work in progress with Dragos Ghioca)

E = elliptic curve / number field K

$$y^2 = x^3 + Ax + B \qquad A, B \in K$$

Néron-Tate (canonical) height function, $P \in E(\bar{K})$

$$\hat{h}_E(P) = \lim_{n \to \infty} \frac{1}{4^n} h_{\text{Weil}}([2^n P]_x)$$

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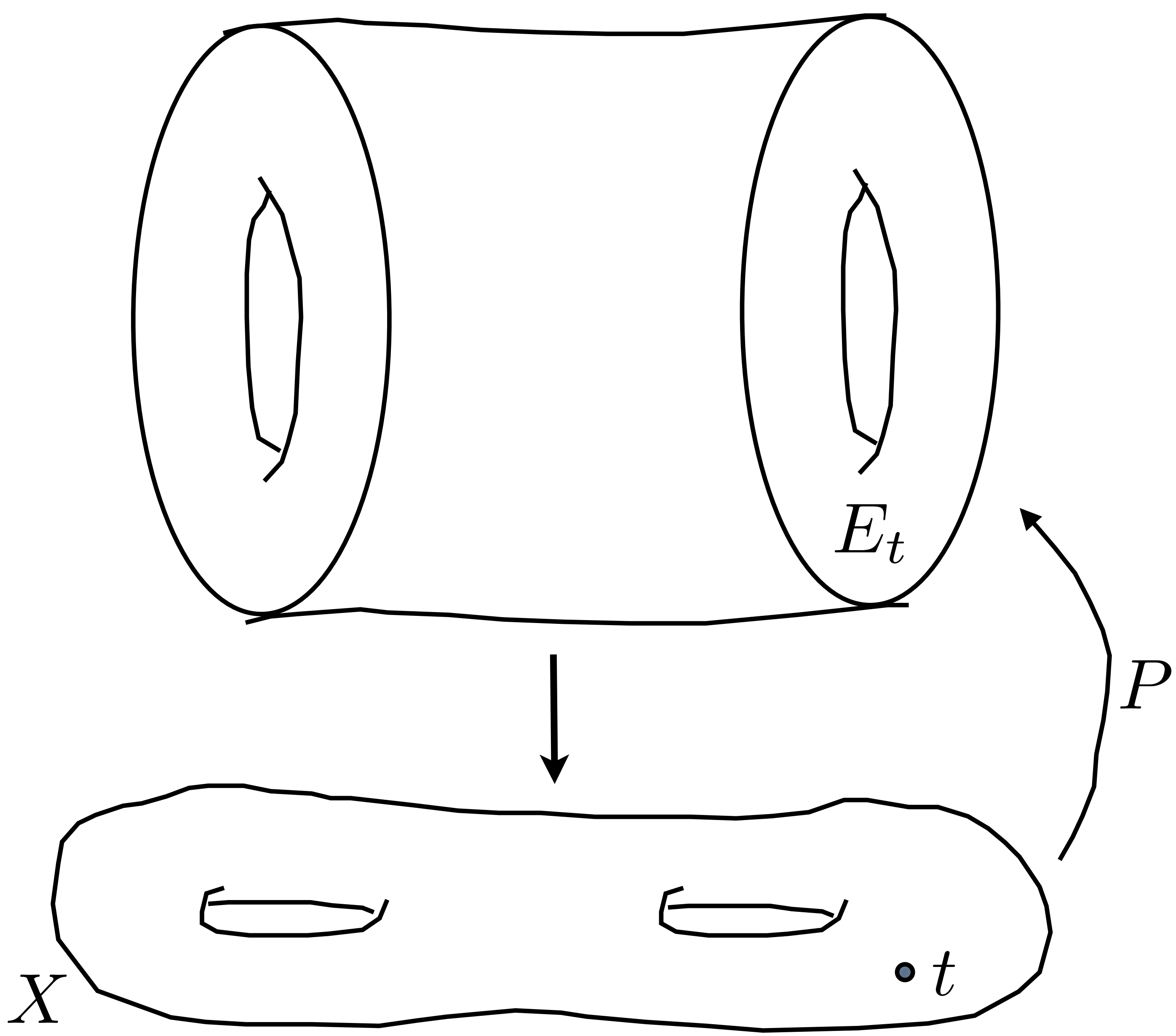

E = elliptic curve / function field k

$$k = K(X)$$

$$P \in E(k)$$

Study $\hat{h}_{E_t}(P_t)$ for $t \in X(\bar{K})$

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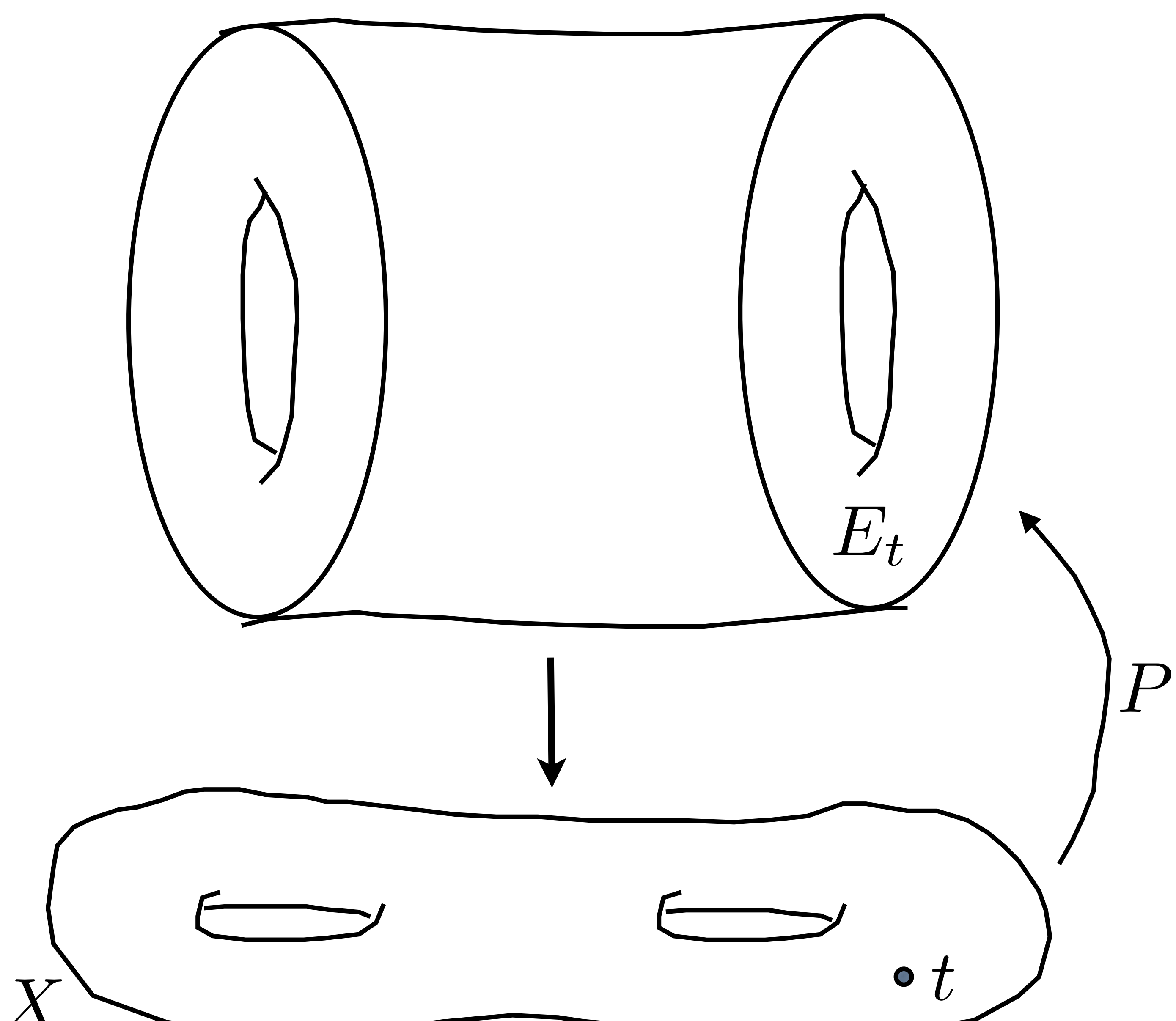

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Theorem. (Silverman, 1983)

$$\lim_{h_X(t)\to\infty} \frac{\hat{h}_{E_t}(P_t)}{h_X(t)} = \hat{h}_E(P)$$

Theorem. (Tate, 1983)

$$\hat{h}_{E_t}(P_t) = h_{X,D_P}(t) + O(1)$$

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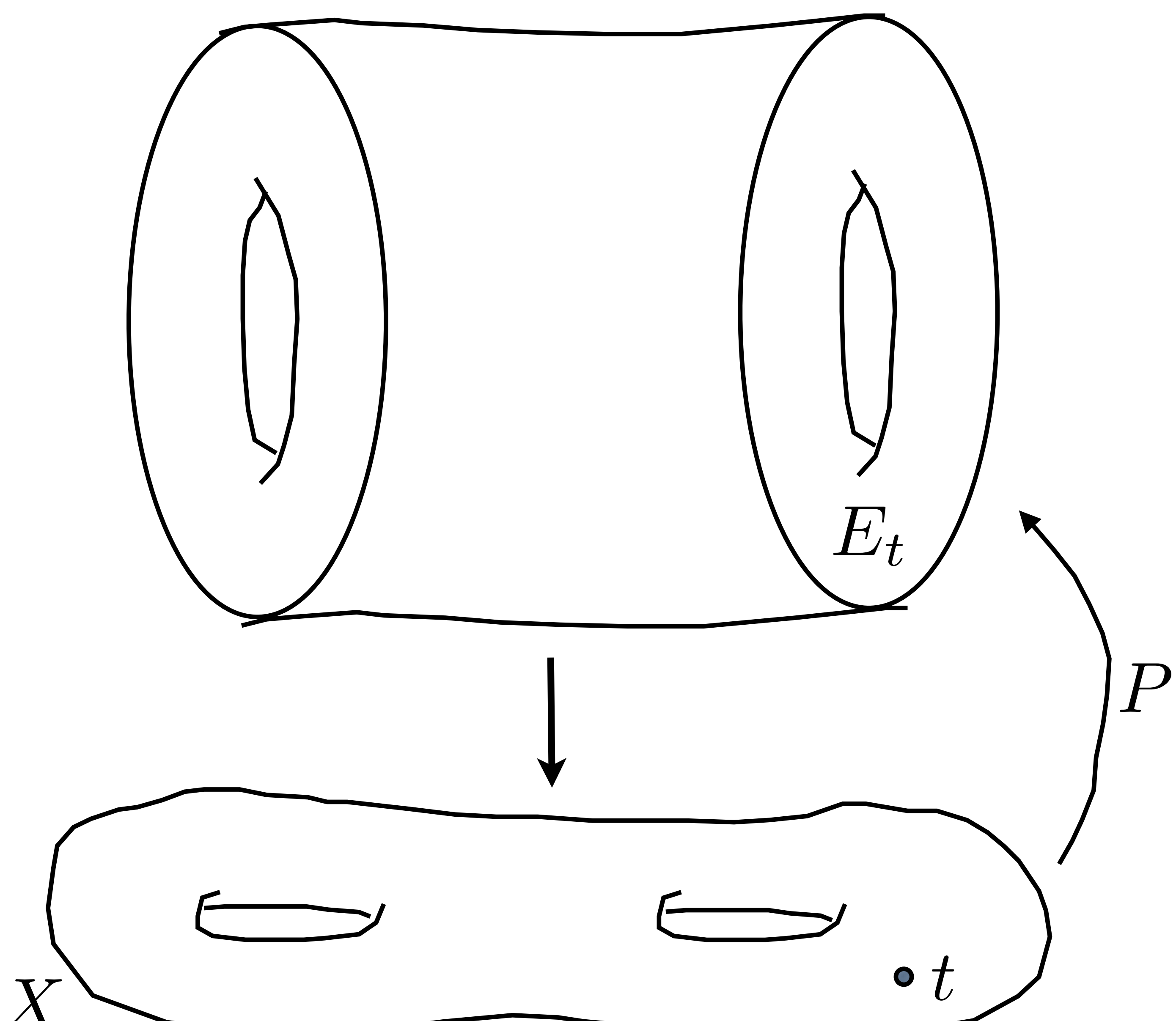

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Silverman's Variation of Canonical Height, 1992-1994

The components in the local decomposition

$$\hat{h}_{E_t}(P_t) = \sum_{v \in M_K} \hat{\lambda}_{E_t,v}(P_t)$$

satisfy

(1) $\hat{\lambda}_{E_t,v}(P_t) = \hat{\lambda}_{E,t_0}(P) \log |u(t)|_v$ + continuous correction term

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$\Longrightarrow$ $\hat{h}_{E_t}(P_t)$ defines a "good height function" on $X(\bar{K})$

i.e., a family of continuous potential functions for adelic measure $\mu = \{\mu_v\}$ on the Berkovich spaces $X^{an}_{\mathbb{C}_v}$, or an adelic metrized line bundle, in sense of Zhang, 1995

$\Longrightarrow$ we are set up to study the distribution of "small" points on X

(e.g. Baker--Rumely, Chambert-Loir, Favre--Rivera-Letelier 2006, Yuan 2008)

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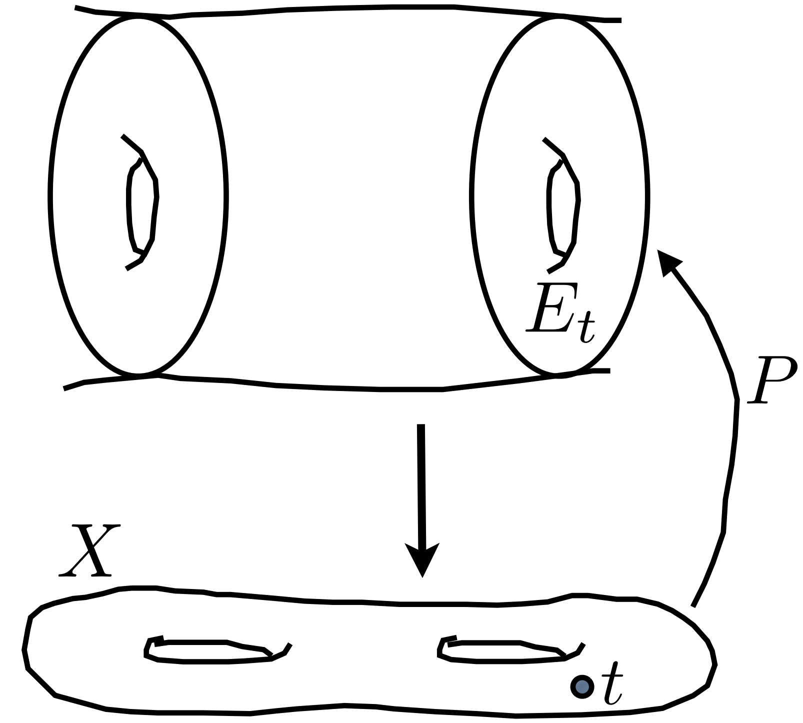
E = elliptic curve / function field $k = K(X)$

$P \in E(k)$



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$$\mu_v = \Delta(\text{correction term})$$

What are these measures on X?
∧
the Berkovich analytification of

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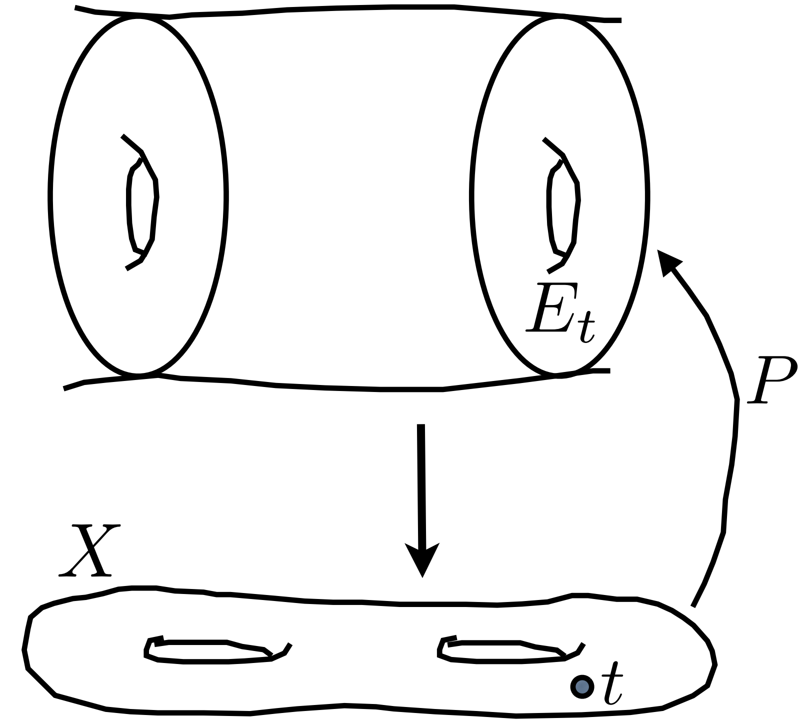

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The measure is a pull-back of the Haar measures on the elliptic curves. This is a special case of the **dynamical bifurcation measure** and the correction term governs the "intensity" of the bifurcation.

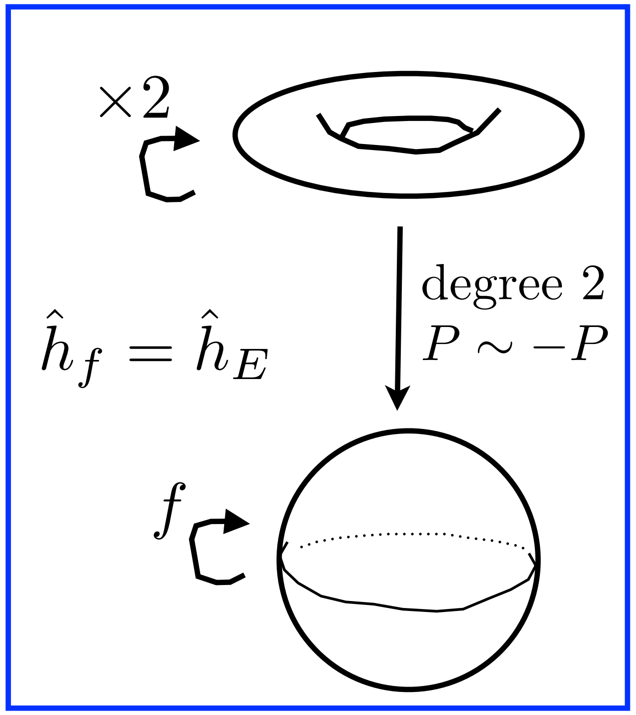
A generalization: Call-Silverman canonical height (1994)

$f : \mathbb{P}^1 \to \mathbb{P}^1 \qquad \deg f > 1$

$\hat{h}_f : \mathbb{P}^1(\bar{K}) \to \mathbb{R}$
determined uniquely by two properties: $\begin{cases} \hat{h}_f(f(z)) = (\deg f)\hat{h}_f(z) \\ \hat{h}_f(z) = h(z) + O(1) \end{cases}$

$$\hat{h}_f(z) = \lim_{n\to\infty} \frac{1}{(\deg f)^n} h(f^n(z))$$
$$= \sum_{v \in M_K} \hat{\lambda}_{f,v}(z)$$

Study variation $\hat{h}_{f_t}(P_t)$ for $t \in X$, in families $\{f_t\}$.



Modern approach:

Extend local heights $\hat{\lambda}_{f_t,v}(P_t)$ to $X^{an}_{\mathbb{C}_v}$.
Take the Laplacian Δ of the local heights, as functions of t.

The variation of the canonical height -- at the archimedean place -- quantifies bifurcations in a traditional dynamical sense.

Example: degree 2 polynomials

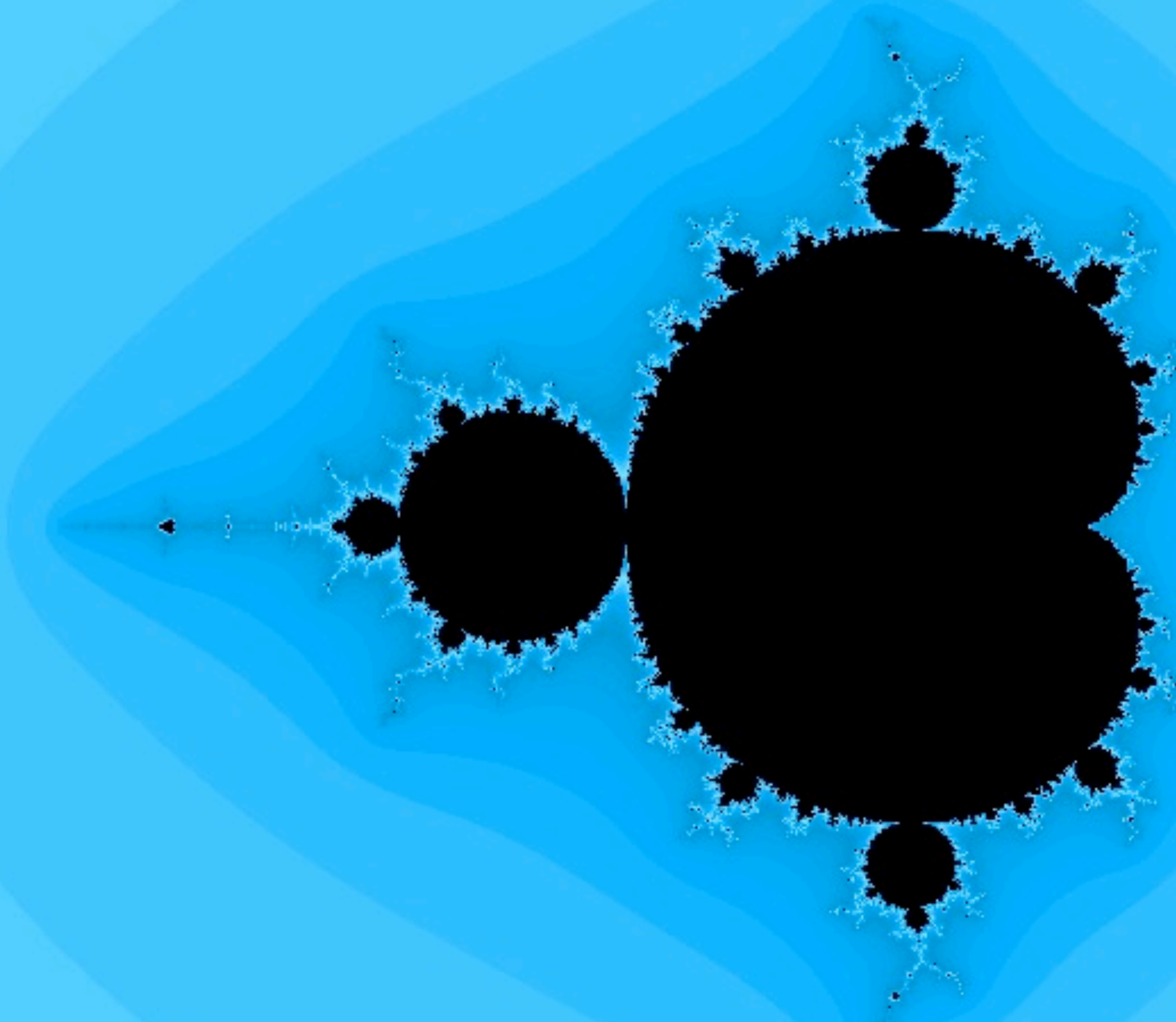
$$f_t(z) = z^2 + t \qquad t \in \mathbb{C}$$

$$P = 0$$

$$\hat{\lambda}_{f_t, v=\infty}(P_t) = \frac{1}{2} \log |t| + \text{correction term}$$

for $|t|$ large

The Mandelbrot set



Bifurcation measure μ_P is harmonic measure on $\partial \mathcal{M}$

(Douady-Hubbard, Sibony 1981, Mañé-Sad-Sullivan 1983)

Example: degree 2 polynomials

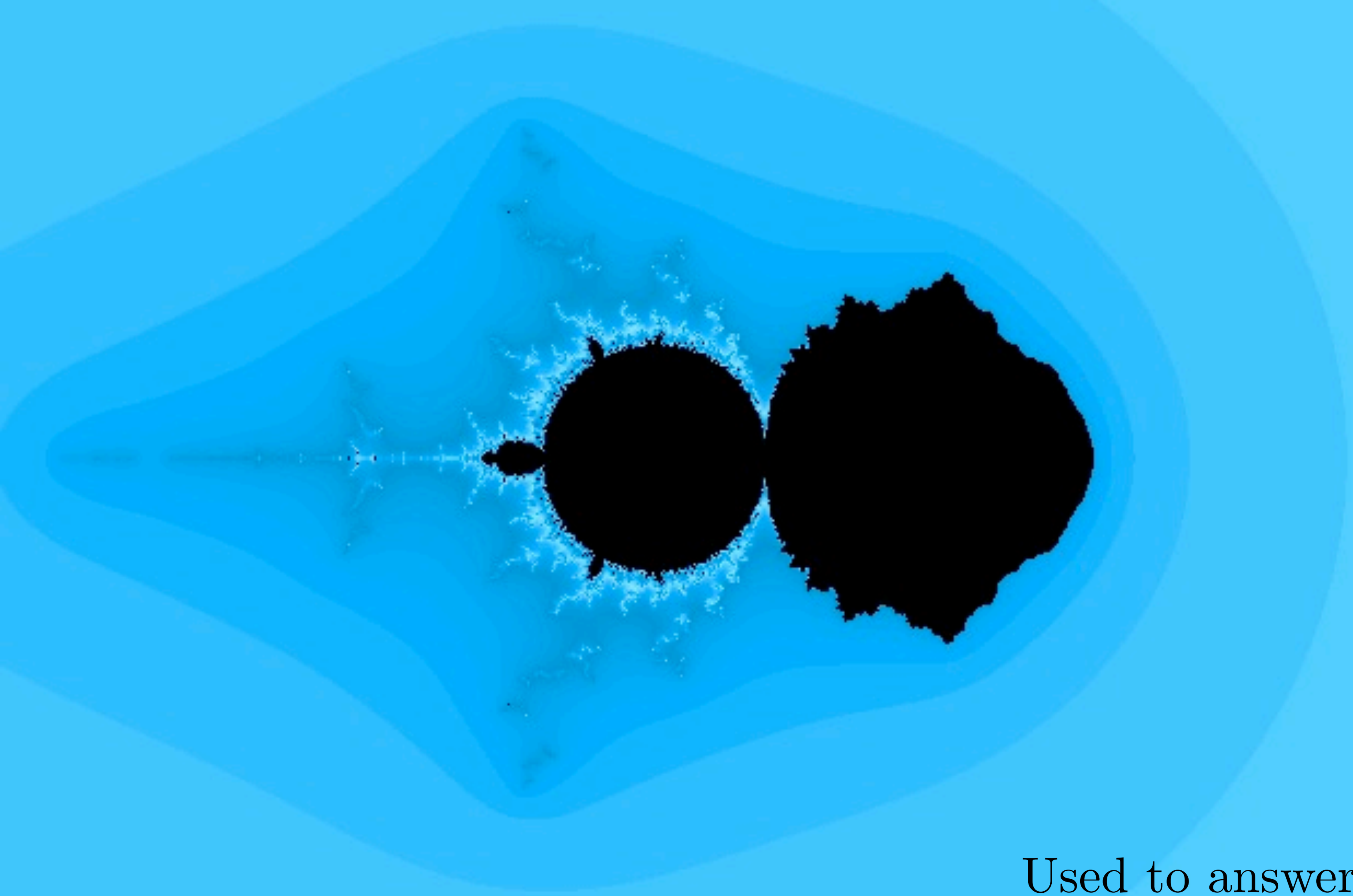
$$f_t(z) = z^2 + t \qquad t \in \mathbb{C}$$

$$P = 1$$

$$\hat{\lambda}_{f_t, v=\infty}(P_t) = \frac{1}{2} \log |t| + \text{correction term}$$

for $|t|$ large

A Mandelbrot-like set



Bifurcation measure μ_P is harmonic measure on $\partial \mathcal{M}$

Used to answer an "unlikely intersections" question posed by Zannier: there are only finitely many t for which both 0 and 1 have finite orbit for f_t.

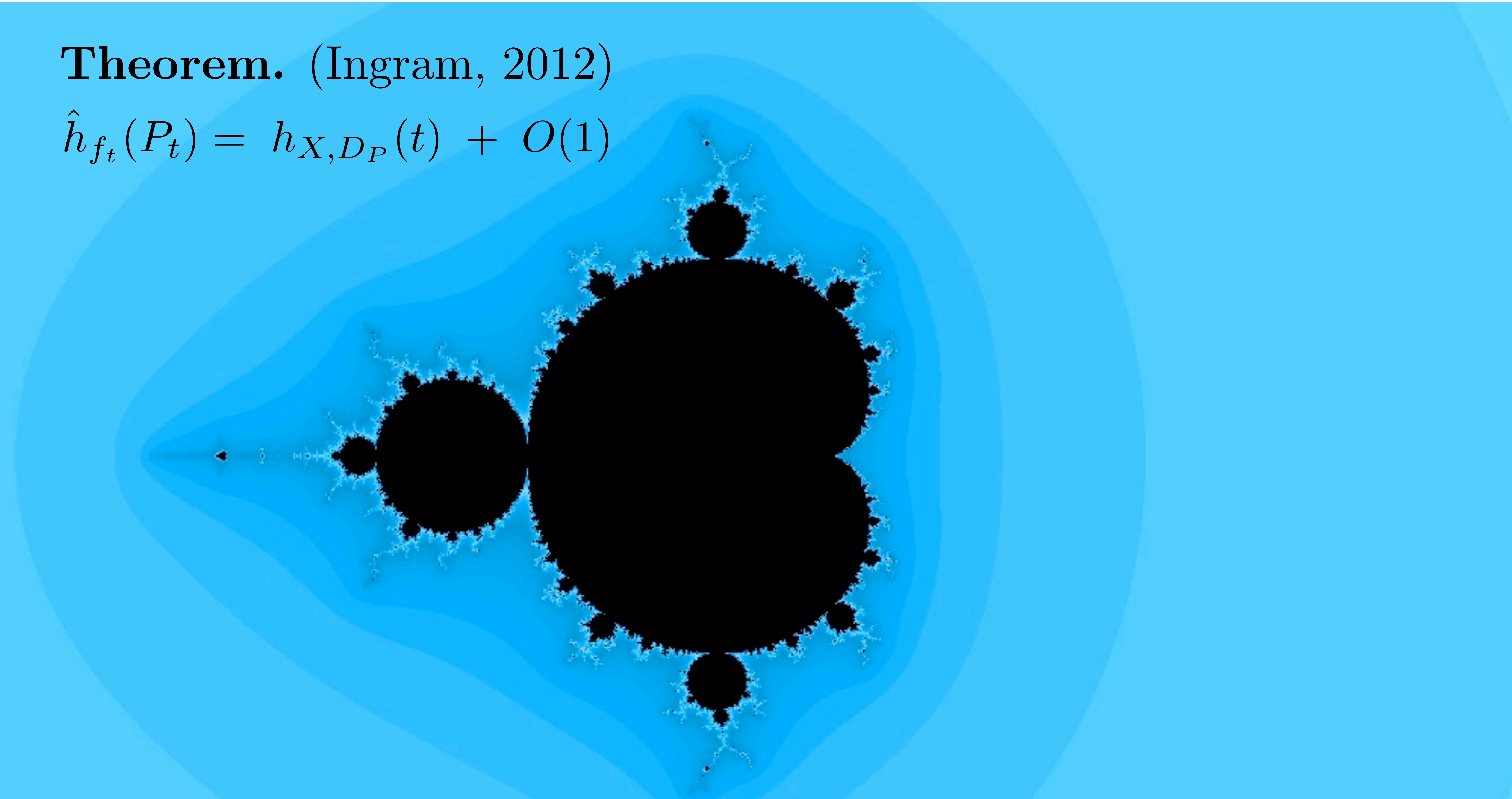
(Baker-D. 2011)

In these examples, the measures are compactly supported (away from point of bad reduction at infinity). So the "correction terms" will be nice harmonic functions near infinity.

For general families of **polynomials**, height functions and measures depend only on rates of escape to infinity. Ingram proved the analog of Tate's 1983 result:

Theorem. (Ingram, 2012)

$$\hat{h}_{f_t}(P_t) = \ h_{X,D_P}(t) \ + \ O(1)$$



Example, in the context of Silverman's Var. Can. Height

$$E_t = \{y^2 = x(x-1)(x-t)\}$$

$$P = (a, \sqrt{a(a-1)(a-t)}) \qquad a \in \mathbb{Q}(t)$$

(1) $\hat{\lambda}_{E_t,v}(P_t) = \hat{\lambda}_{E,t_0}(P) \log|u(t)|_v$ + continuous correction term

(2) correction term $\equiv 0$ for all but finitely many $v \in M_K$

$$\mu_v = \Delta(\text{correction term})$$

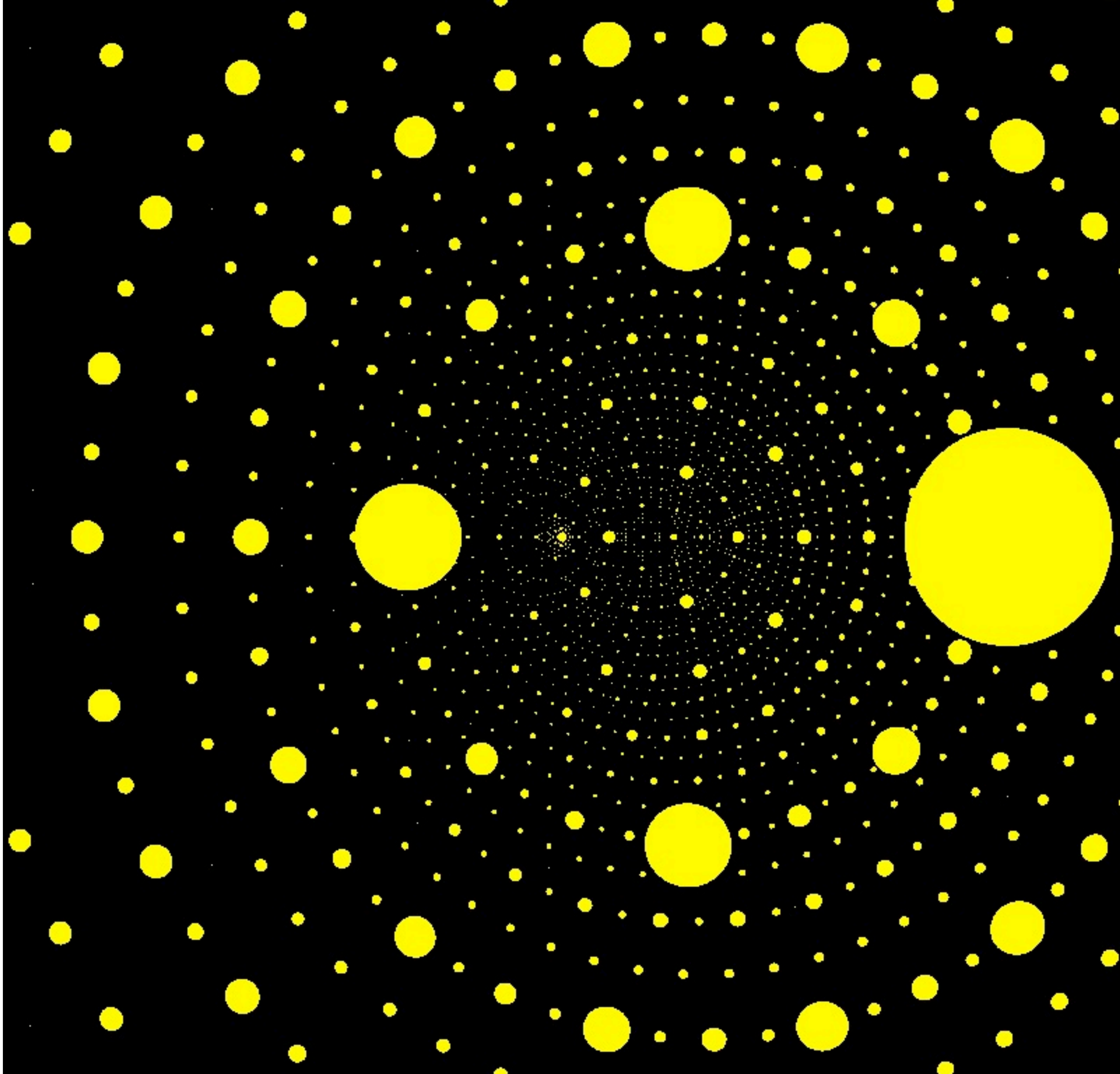
Fact 1. The parameters $t \in X$ where P_t is torsion on E_t are equidistributed with respect to these measures $\mu_{P,v}$.

Fact 2. The measures $\{\mu_{P_v}\}$ coincide with $\{\mu_{Q_v}\}$ if and only if the points P and Q are linearly related on E.
This can be seen already at the archimedean place.

(D.-Wang-Ye, 2015)
building on the results of (Masser-Zannier, 2008, 2010, 2012),
applying equidistribution theorems on Berkovich P^1
(Baker-Rumely, Chambert-Loir, Favre-Rivera-Letelier, 2006)

$a = 2$

Plot: parameters t where a is the x-coordinate of a torsion point on E_t, of order 2^n with $n < 8$.

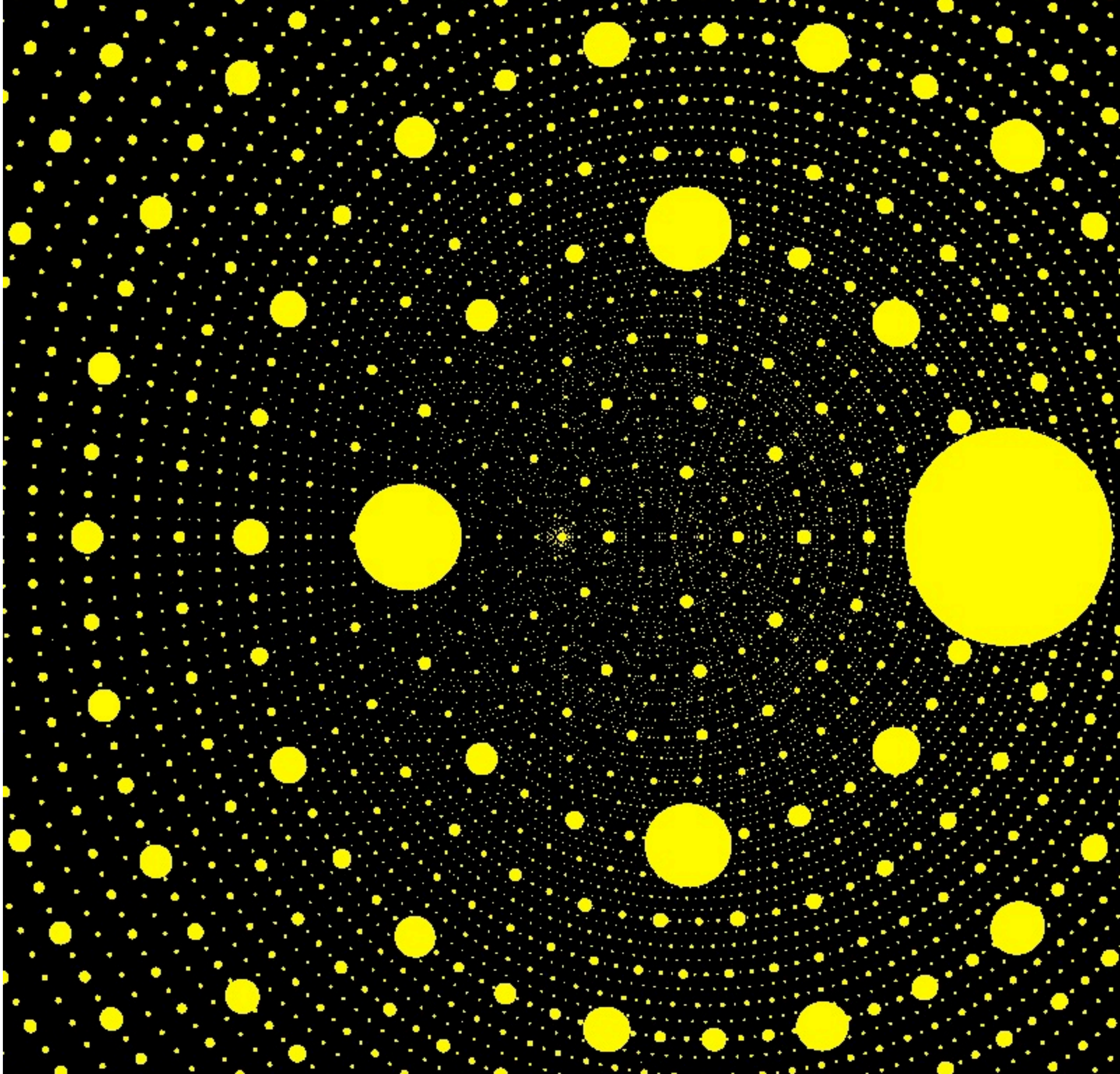


$-3 < \operatorname{Re} t < 5$

$-4 < \operatorname{Im} t < 4$

$a = 2$

Plot: parameters t where a is the x-coordinate of a torsion point on E_t, of order 2^n with $n < 10$.



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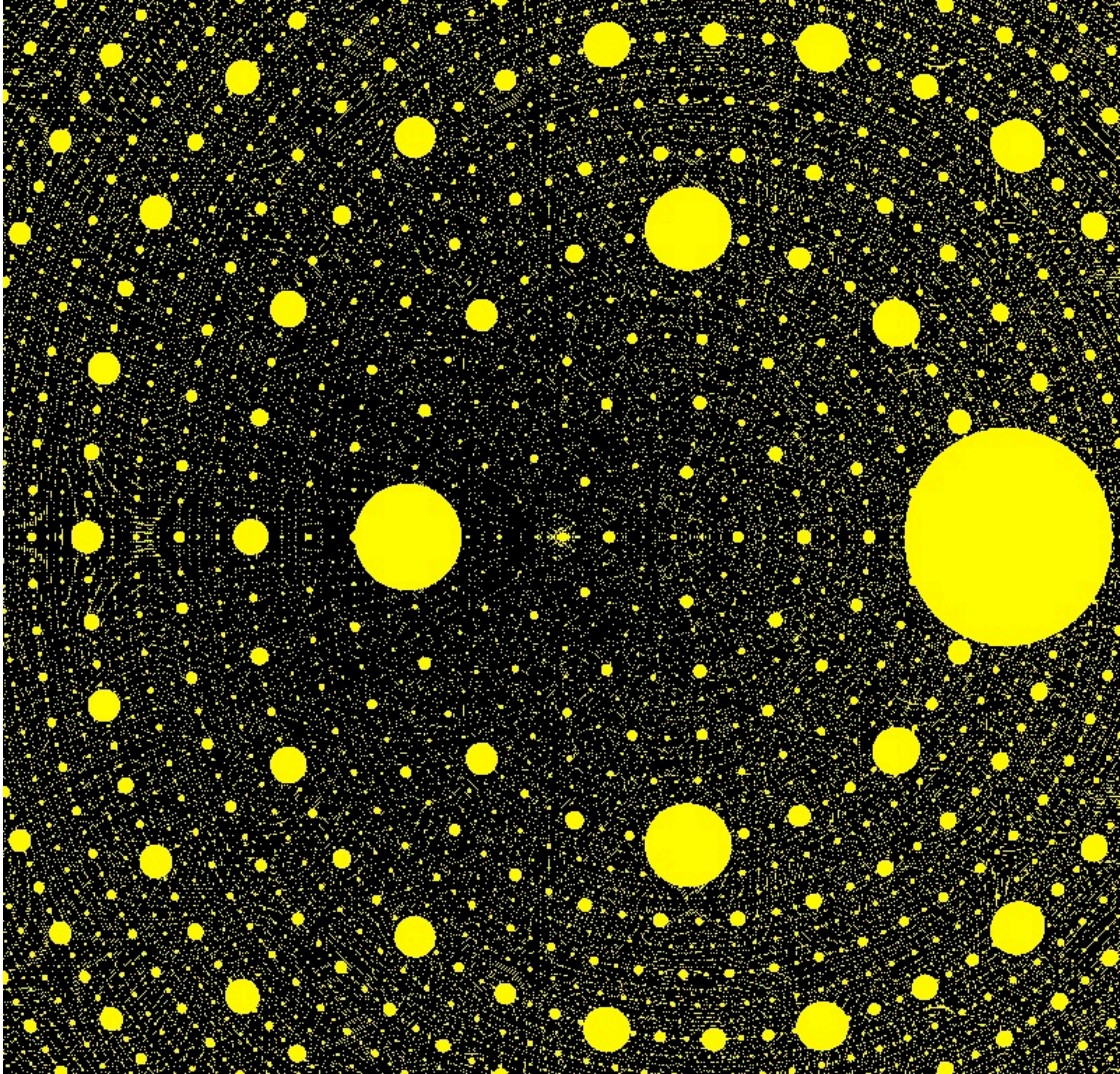
Plot: parameters t where a is the x-coordinate of a torsion point on E_t, of order 2^n with $n < 15$.



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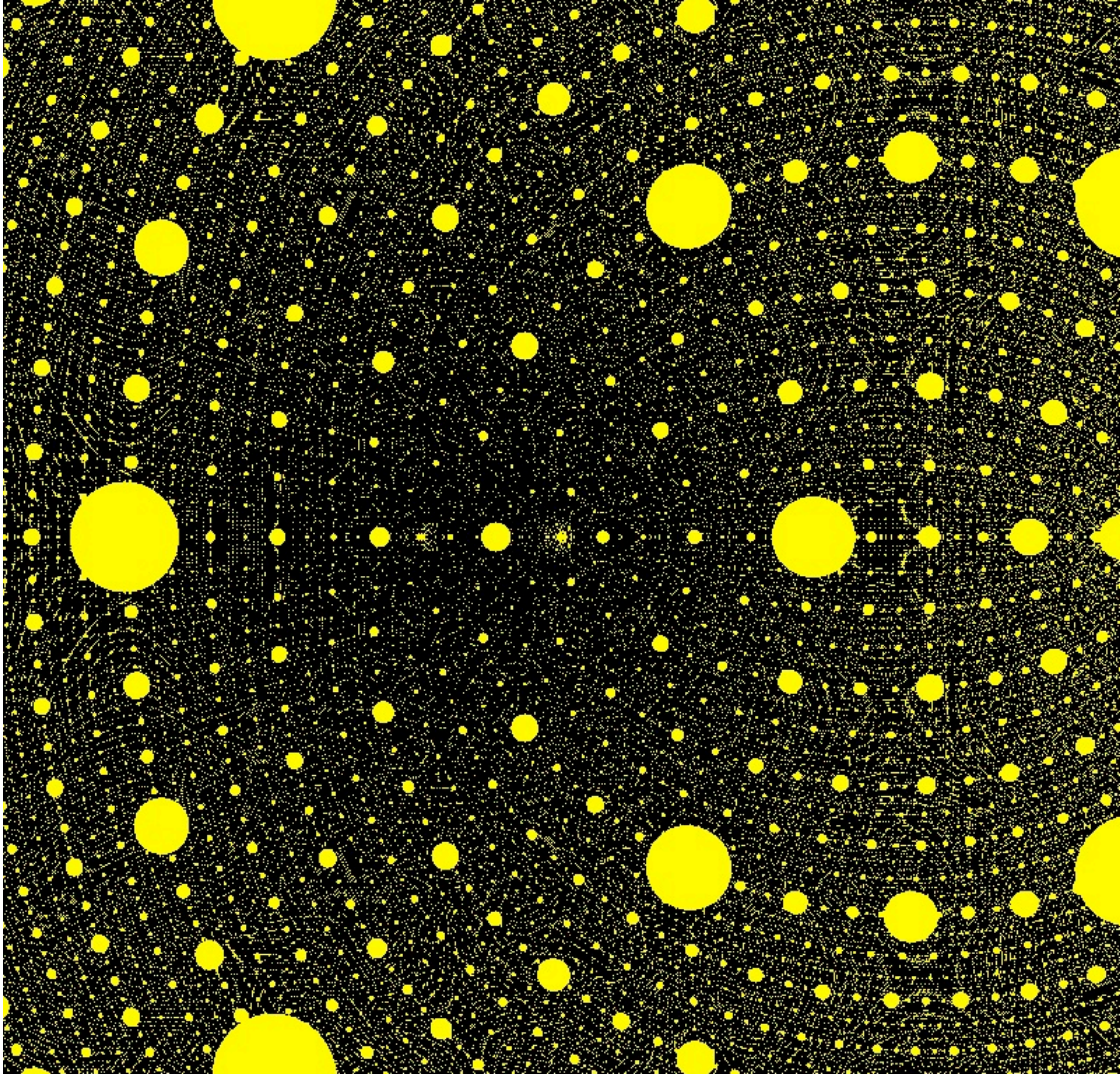

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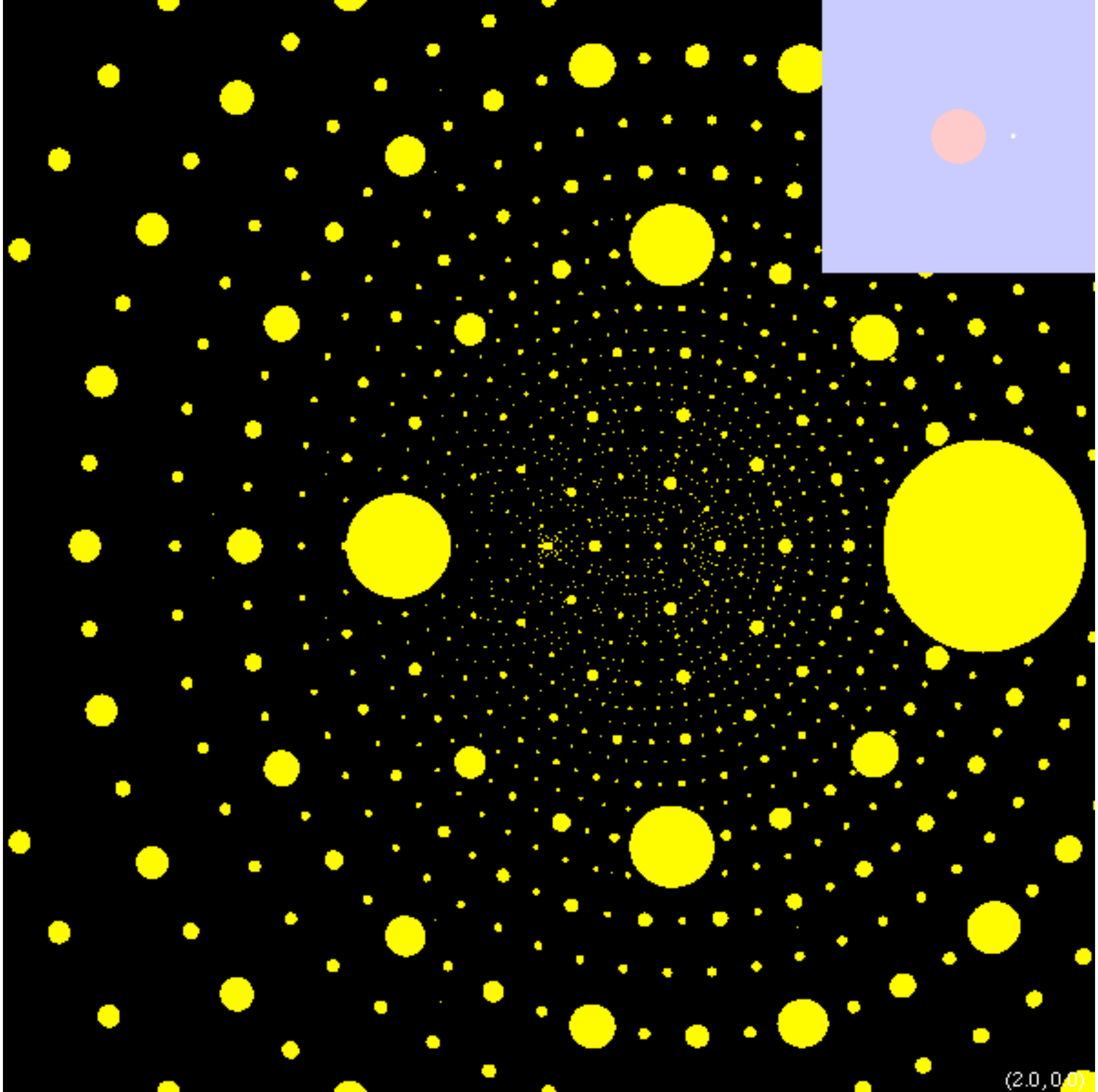
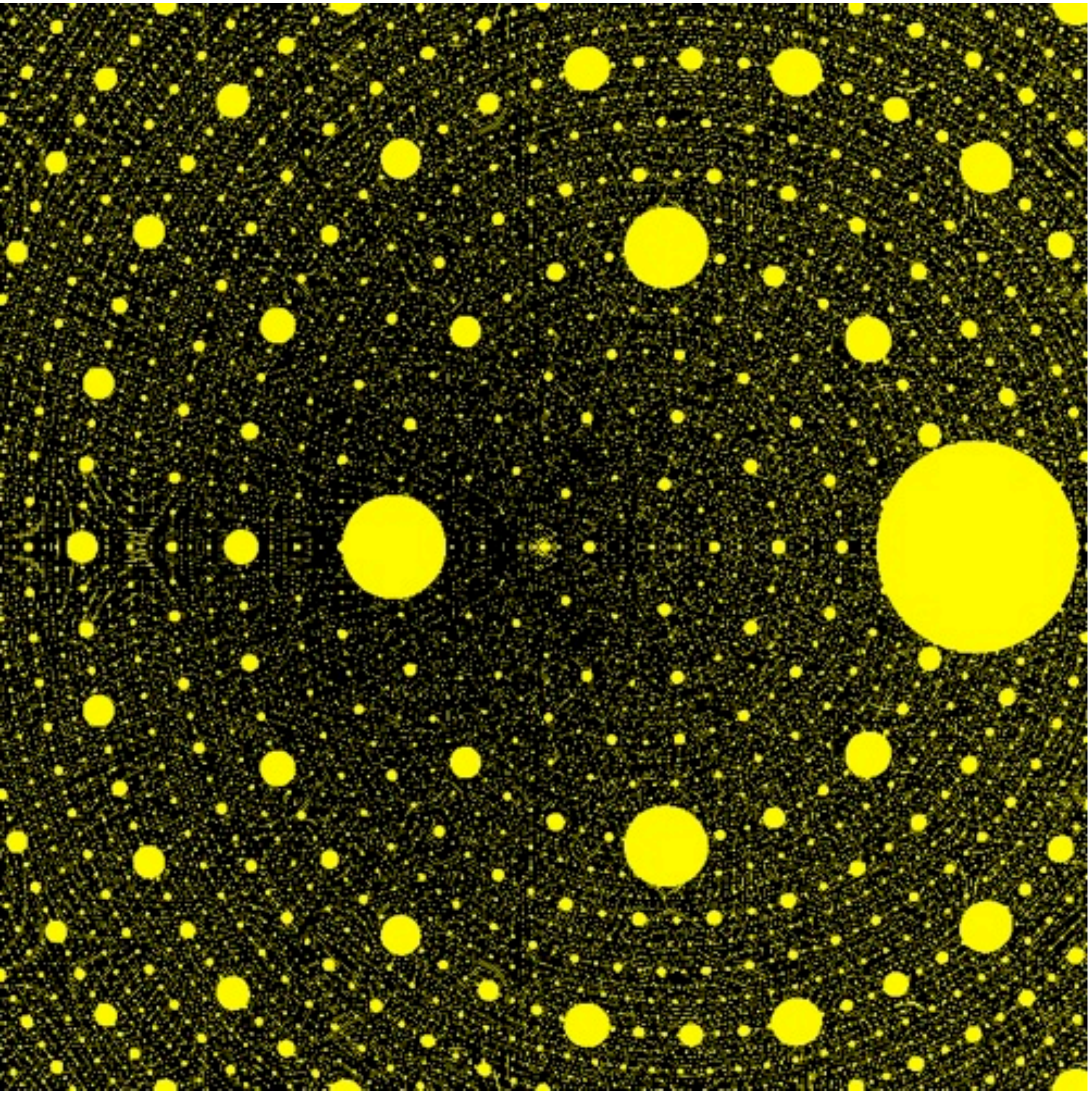
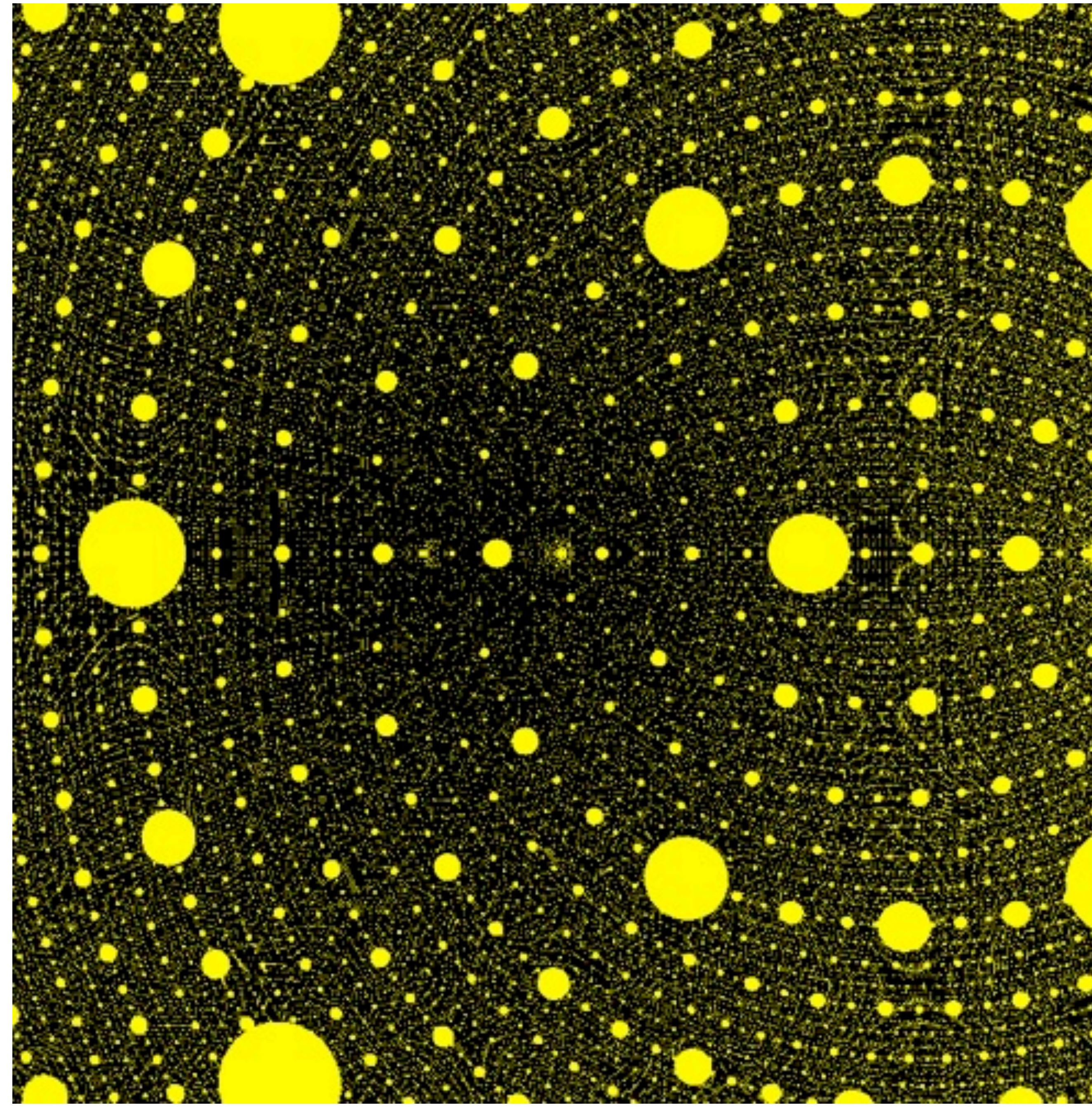

(2.0,0.0)

$a = 2$

$\mu_a = \mu_b$ if and only if $a = b$

$b = 5$





$E_t = \{y^2 = x(x-1)(x-t)\}$

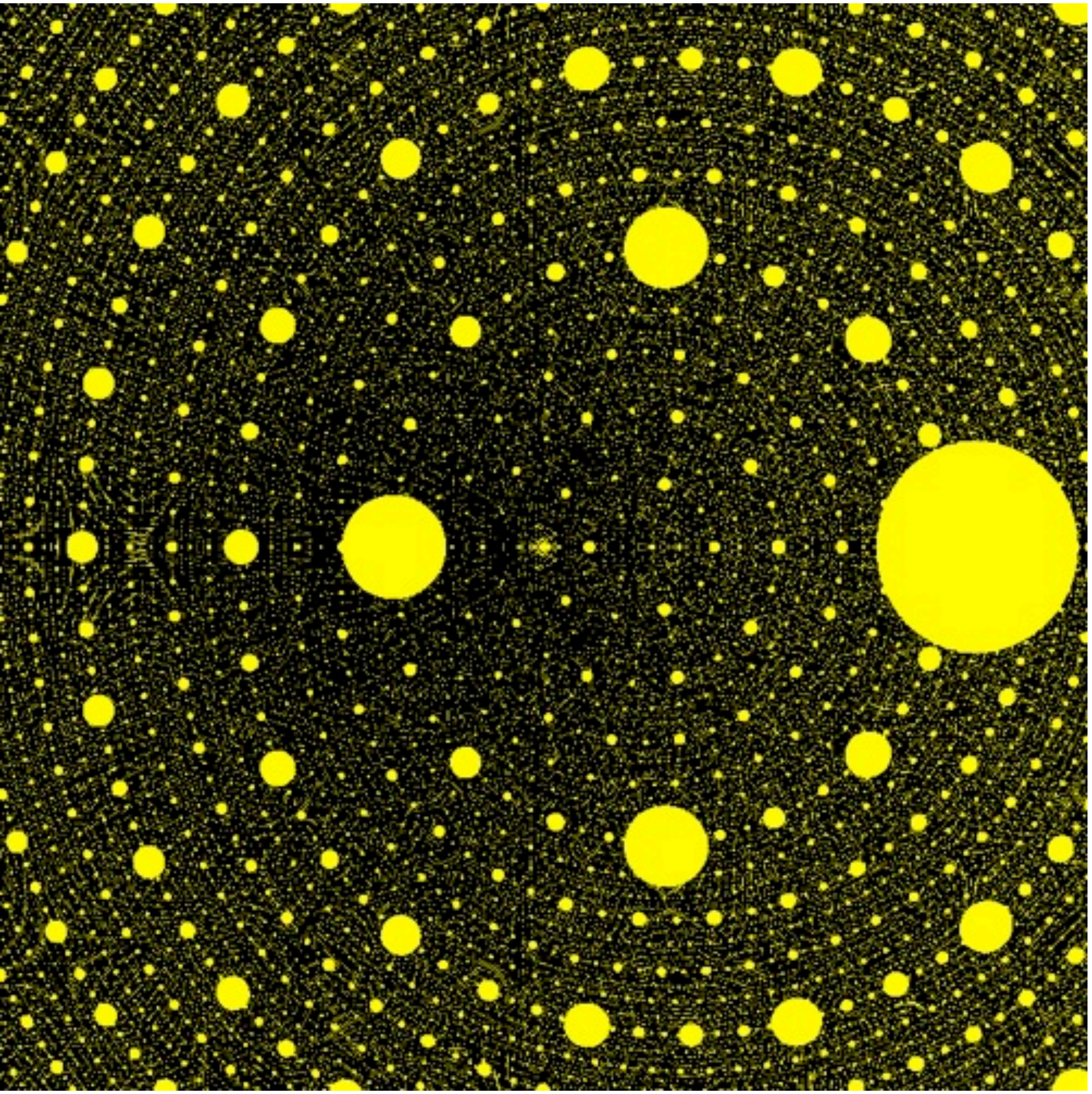
The Haar measure on E_t pushed down to $\mathbb{P}^1$ is

$$\mu_t = \frac{C(t)}{|z(z-1)(z-t)|}\,|dz|^2 \quad \text{where } C(t) = 2|t(t-1)|\rho_\Sigma(t).$$

Density for hyperbolic metric on triply-punctured sphere (McMullen)

$\mu_a = \mu_b$ if and only if $a = b$

$a = 2$



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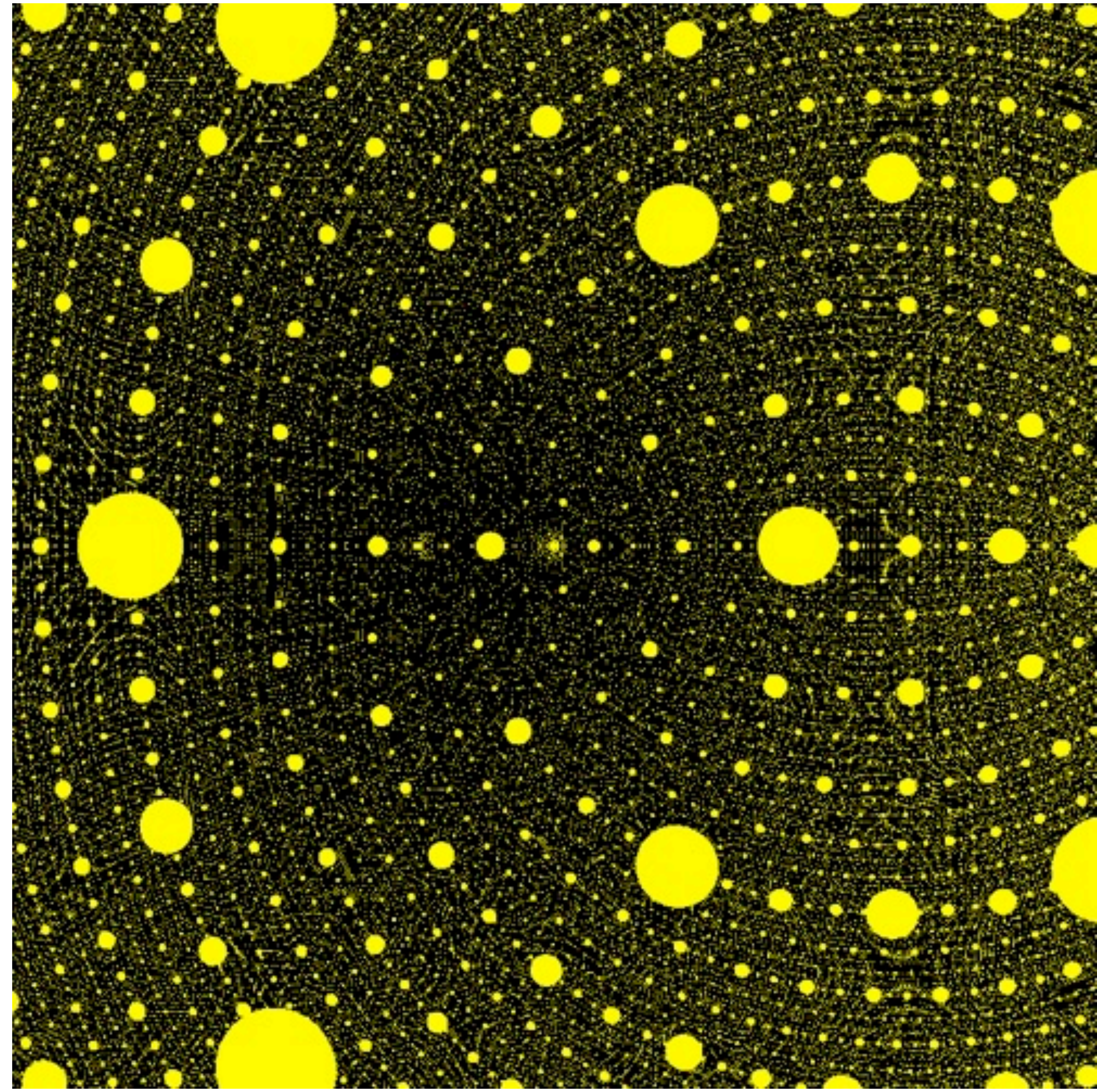

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Potential function for μ_t: $\Longrightarrow$ Potential function for μ_a:

$$g_t(z) = 2C(t) \int_{\mathbb{P}^1} \frac{\log|z-\zeta|}{|\zeta(\zeta-1)(\zeta-t)|} \, |d\zeta|^2$$

$$g_a(t) = 2C(t) \int_{\mathbb{P}^1} \frac{\log|a-\zeta|}{|\zeta(\zeta-1)(\zeta-t)|} \, |d\zeta|^2$$

K = number field

E = elliptic curve / function field $k = K(X)$

$P \in E(k)$

Theorem. (Silverman) The components in the local decomposition

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for t near $t_0 \in X(\bar{K})$.

What do we know for dynamical canonical height?

Known for: Latt`es maps (a corollary of above)
Particular families of polynomials and rational maps
(Baker-D., D.-Wang-Ye, Ghioca-Hsia-Tucker, Ghioca-Mavraki, Ingram)

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False for general dynamical families! (D.-Wang-Ye, 2015)

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A more basic question:
do we understand the leading terms
for general dynamical families?

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A more basic question:
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Fact. $\hat{h}_E(P)$ and $\hat{\lambda}_{E,t_0}(P)$ are rational numbers. (Tate, Manin)

Explanation. These are intersection numbers on a Néron model.

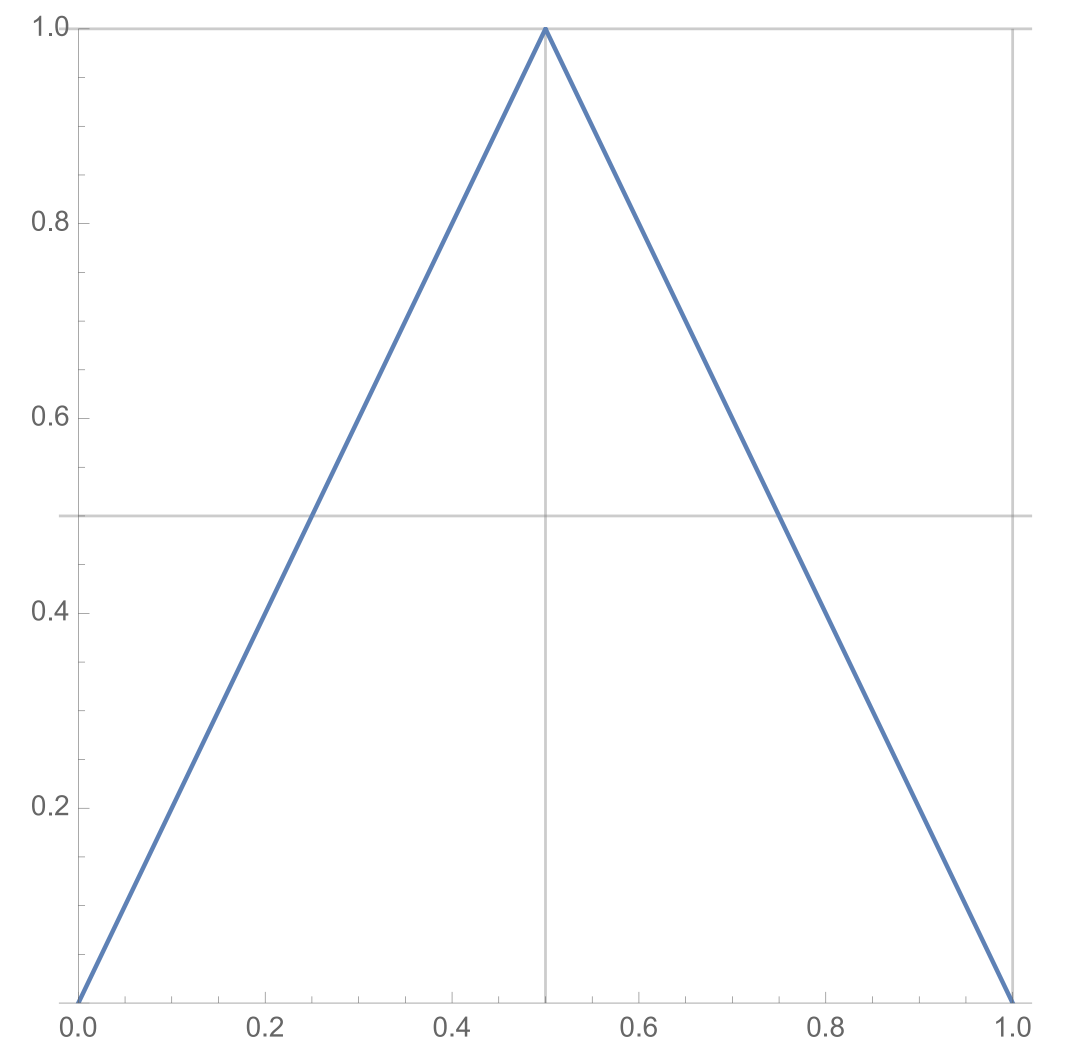
Another Fact. The analogous "weak" Néron models do not always exist in the dynamical setting. (Call-Silverman, Hsia)

Rationality of canonical height

(work in progress with Dragos Ghioca)

1. There is a dynamical proof that canonical heights are always rational for elliptic curves. (True locally at non-archimedean places, true globally for function fields.)

Idea:
Dynamics of the multiplication-by-2 map on E, on the Berkovich $\mathbf{P}^1$.
Julia set is an interval.
Action is by the tent map of slope 2, all rational points are preperiodic.
(Favre--Rivera-Letelier)



The tent map of slope 2

Compare:

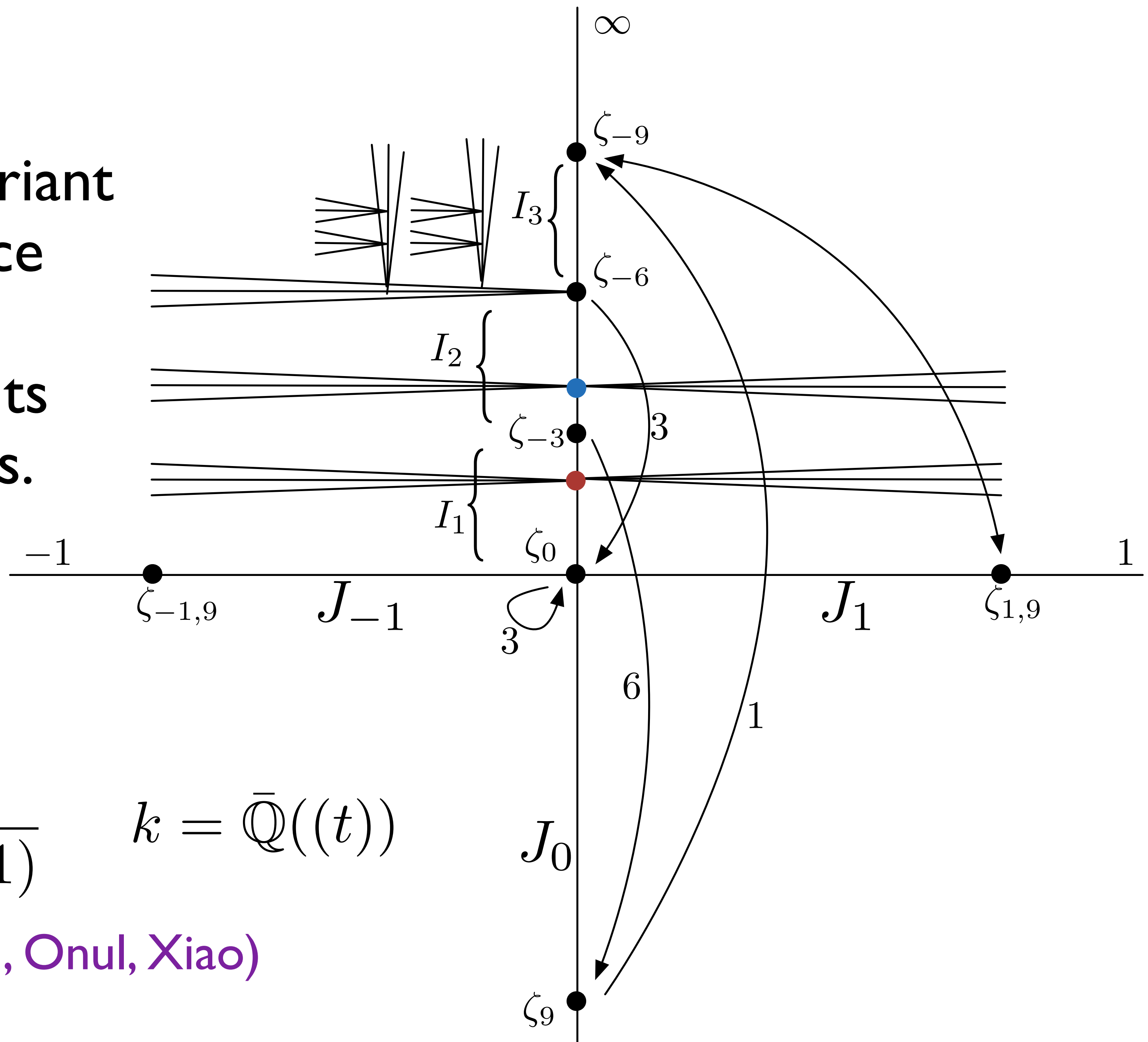
Theorem. (Ingram, 2012)
For polynomials, local heights at non-archimedean places are rational.

Rationality of canonical height

(work in progress with Dragos Ghioca)

2. There exist dynamical systems on P^1 and points with irrational local heights!

Idea:
Julia set contains forward invariant intervals in the Berkovich space AND classical points.
There are Cantor sets of points containing aperiodic itineraries.



$$f_t(z) = \frac{t^{18}z^6 + 1}{t^{18}z^6 + z(z-1)(z+1)} \qquad k = \bar{\mathbb{Q}}((t))$$

(Bajpai, Benedetto, Chen, Kim, Marschall, Onul, Xiao)

Rationality of canonical height

(work in progress with Dragos Ghioca)

2. There exist dynamical systems on P^1 and points with irrational local heights! BUT, we expect these points to be transcendental...

(Fatou, Bell-Bruin-Coons, Adamczewski-Bell)

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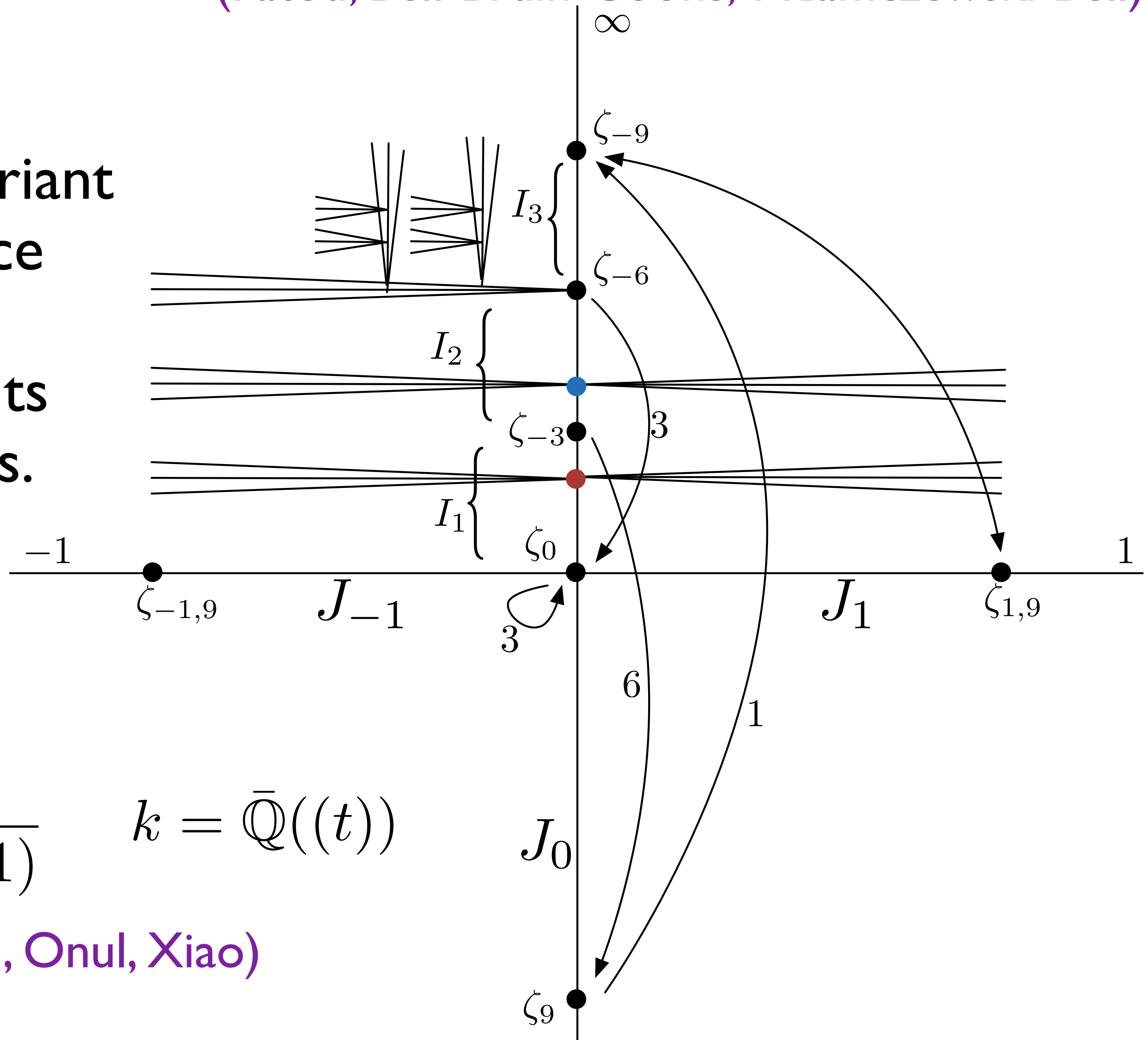

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Next steps

$$k = K(X),\ f : \mathbb{P}^1 \to \mathbb{P}^1 \text{ defined over } k,\ \ P \in \mathbb{P}^1(\bar{k})$$

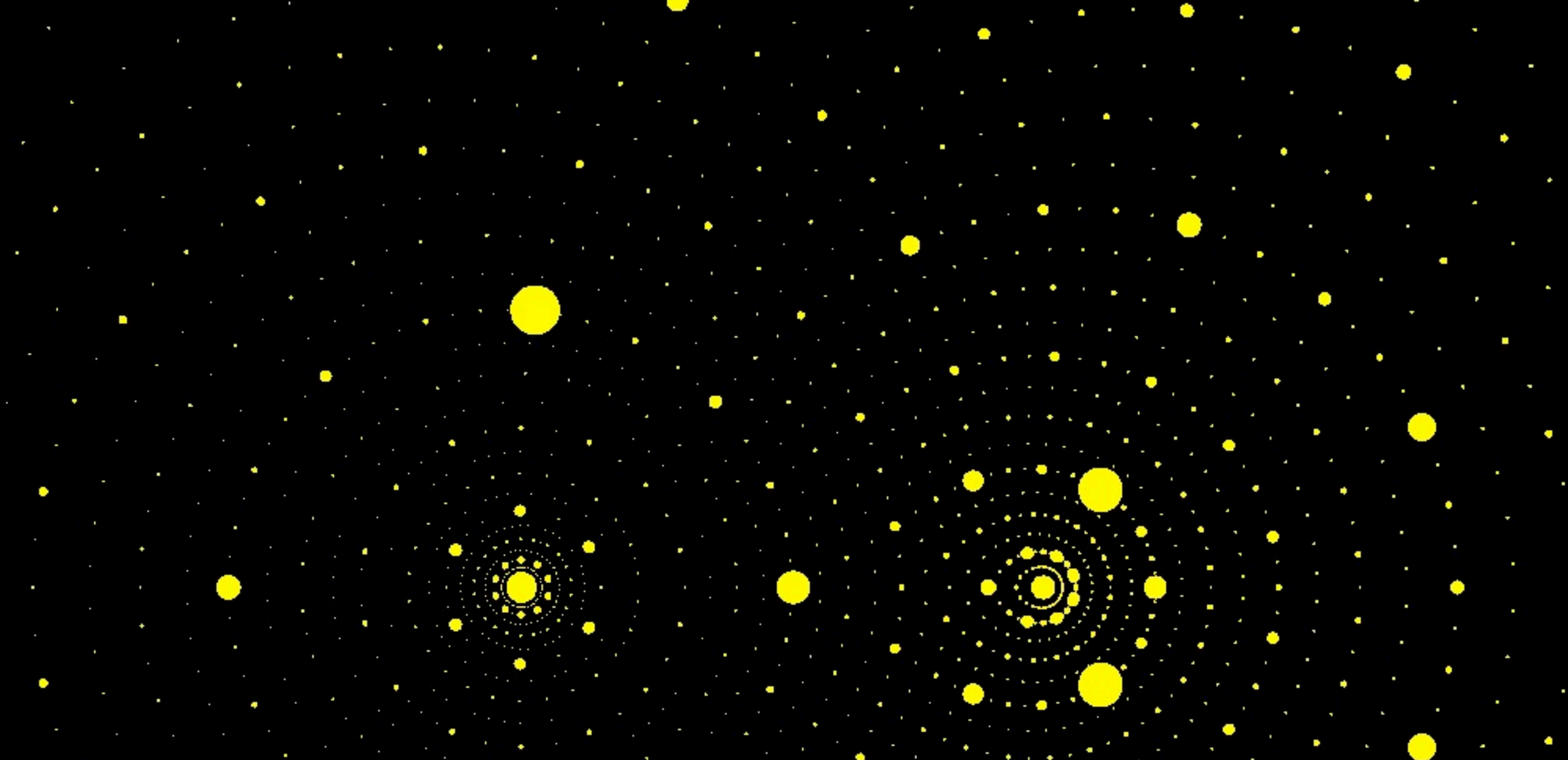
Question. Is the canonical height $\hat{h}_f(P)$ rational?

Question. Is there a good intersection-theoretic description of $\hat{h}_f(P)$, even in the absence of (weak) Néron models?

Question. Is there a divisor $D_P \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ so that

$$\hat{h}_{f_t}(P_t) = \ h_{X,D_P}(t)\ +\ O(1)$$

Question. Are the pieces in the local decomposition of $\hat{h}_{f_t}(P_t)$ "nice" functions of t?



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