Zariski Main Theorem for Henselian Rigid Spaces

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Contents

Henselian rigid geometry

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- "Usual" rigid geometry:

$$\boxed{\text{formal scheme } X} \xrightarrow{\text{rig}} \boxed{\text{rigid space } X^{\text{rig}}}$$

Henselian rigid geometry

- "Usual" rigid geometry:

formal scheme X	$\stackrel{\text{rig}}{\longmapsto}$	rigid space X^{rig}

- Henselian rigid geometry:

henselian scheme X	$\stackrel{\text{rig}}{\longmapsto}$	rigid space X^{rig}

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- So one can say that hRG gives more "pro-algebraic" hybrid between algebraic geometry and analytic geometry.

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  - Application: Prolongation of flat families of closed subspaces in a projective variety.
    - Further application to the construction of "analytic Chow groups" (forthcoming work by M. Kerz & S. Saito).

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Definition

- A is said to be *I-adically henselian*
  $\overset{\text{def}}{\Leftrightarrow}$ any étale morphism $\phi\colon X \to S$ with $\phi^{-1}(D) \cong D$ admits a section.
  $\Leftrightarrow$ $I \subseteq \operatorname{Jac}(A)$, $\forall$monic $F(T) \in A[T]$ with

$$F(0) \equiv 0 \bmod I \text{ and } F'(0) \in (A/I)^{\times},$$

  $\exists a \in I$ such that $F(a) = 0$.

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Henselization

- $A \to A^h$ is flat, and is faithfully flat if $I \subseteq \operatorname{Jac}(A)$.

Henselian finite type algebras

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- A *henselian finite type* V-algebra is a V-algebra isom. to $V\{X_1, \ldots, X_n\}/\mathfrak{a}$.

- If $A = V\{X_1, \ldots, X_n\}/\mathfrak{a}$ is V-flat, then $\mathfrak{a}$ is finitely generated.
  - If, moreover, V is of height 1, then A is what we should call "henselian admissible" V-algebra.

Henselian affinoid algebras

- V: a-adically separated a-adically henselian valuation ring ($a \in \mathfrak{m}_V \setminus \{0\}$),
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$$K\{X_1, \ldots, X_n\} \overset{\text{def}}{=} V\{X_1, \ldots, X_n\} \otimes_V K$$
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$$\mathcal{A} = K\{X_1, \ldots, X_n\}/\mathfrak{a}$$

by an ideal $\mathfrak{a} \subseteq K\{X_1, \ldots, X_n\}$.

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For any henselian affinoid algebra $\mathcal{A}$ over K, there exists an *injective finite* *morphism*

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Lemma

Let $A \to B$ be a finite type morphism of rings, and $I \subseteq A$ an ideal. If $A/I \to B/IB$ is finite, then so is $A^h \to B^h$.

- Classical (refined) ZMT: finiteness extends to an étale neighborhood.

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Henselian affinoid algebras are Jacobson.

Contents

Henselian schemes: References

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Henselian spectrum

$\mathrm{Sph}\, A =$ top. locally ringed space by

- the set of all open prime ideals of A
- with the subspace topology induced from the Zariski topology of $\mathrm{Spec}\, A$;
- for any $f \in A$, $\mathcal{D}(f) = D(f) \cap X$ and $\mathscr{O}_X(\mathcal{D}(f)) = (A_f)^h$, which gives a sheaf of top. rings on X.

Henselian schemes

Henselian schemes

Definition

- An *affine henselian scheme* is a top. loc. ringed space isom. to $(X = \mathrm{Sph}\, A, \mathscr{O}_X)$ for an I-adically henselian ring A.
- A top. locally ringed space $(X, \mathscr{O}_X)$ is called a *henselian scheme* if it is covered by affine henselian schemes.

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- $A \mapsto \operatorname{Sph} A$ gives duality between the cat. of henselian rings with cont. homomorphisms and the cat. of affine henselian schemes.
- There is the notion of *henselization* $X^h|_Y$ of schemes along closed subschemes.

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- **CHs*** = cat. of coherent (= q-cpt & q-sep) henselian schemes with adic morphisms.

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- $\mathbf{CHs}^*$ = cat. of coherent (= q-cpt & q-sep) henselian schemes with adic morphisms.

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- First appearance in literature:
  - Fujiwara, K.: *Theory of tubular neighborhood in étale topology*. Duke Math. J. **80** (1995), no. 1, 15–57.

Visualization and points

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Zariski-Riemann space

For a coherent henselian rigid space $\mathscr{X} = X^{\mathrm{rig}}$,

$$\langle \mathscr{X} \rangle = \varprojlim_{X' \to X} X',$$

where X' runs over all adm. blow-ups of X.

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- $\langle \mathscr{X} \rangle$ is sober and coherent.
- Points of $\langle \mathscr{X} \rangle$ are in bijection with the equiv. classes of rigid points, i.e., morphisms of the form

$$(\mathrm{Sph}\, V)^{\mathrm{rig}} \longrightarrow \mathscr{X},$$

  where V is an a-adically sep. and henselian valuation ring.

Contents

GAGA

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Situation

- V: a-adically sep. and henselian valuation ring, $K = V[\frac{1}{a}]$.
- A: *henselian finite type* V-algebra.
  $\rightsquigarrow \mathcal{A} = A[\frac{1}{a}]$: henselian affinoid algebra over K.
- $U = \operatorname{Spec} \mathcal{A} \subseteq S = \operatorname{Spec} A$, $D = \operatorname{Spec} A/aA$,
- $\mathscr{S} = (\operatorname{Spf} A)^{\mathrm{rig}}$: the affinoid associated to $\mathcal{A}$.

GAGA

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- $U = \operatorname{Spec} \mathcal{A} \subseteq S = \operatorname{Spec} A$, $D = \operatorname{Spec} A/\mathfrak{a}A$,
- $\mathscr{S} = (\operatorname{Spf} A)^{\mathrm{rig}}$: the affinoid associated to $\mathcal{A}$.

The *GAGA functor*

$$\left\{\begin{matrix}\text{sep. of finite type} \\ \text{schemes over } U\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{locally of finite type} \\ \text{henselian rigid spaces} \\ \text{over } \mathscr{S}\end{matrix}\right\}$$
$$X \longmapsto X^{\mathrm{an}}.$$

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<u>GAGA theorems</u> $\Longleftarrow$ GHGA theorems

Affinoid valued points

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- Any $\alpha\colon \mathscr{T} = (\operatorname{Sph} B)^{\mathrm{rig}} \to X^{\mathrm{an}}$ from a finite type affinoid canonically corresponds to a mor. $\tilde{\alpha}\colon s(\mathscr{T}) = \operatorname{Spec} B[\frac{1}{a}] \to X$ of schemes.

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Theorem

$$\left\{\begin{array}{l}\text{morphism}\\ \alpha\colon \mathscr{T} \to X^{\mathrm{an}}\\ \text{of rigid spaces}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{l}\text{pair } (\beta, h) \text{ consisting of } \beta\colon \mathscr{T} \to \mathscr{S} \text{ and}\\ h\colon s(\mathscr{T}) \to X \text{ such that the diagram}\\ \begin{array}{ccc} s(\mathscr{T}) & \xrightarrow{h} & X \\ & {\scriptstyle s(\beta)}\searrow & \downarrow{\scriptstyle f} \\ & & U \end{array} \text{ commutes}\end{array}\right\}$$

$$\alpha \longmapsto (f^{\mathrm{an}} \circ \alpha, \tilde{\alpha})$$

- Cf. [FK, Chap. **II**, Theorem 9.2.2].

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Definition

- A henselian rigid space $\mathscr{Z}$ is said to be *point-like* if it is coherent and reduced, having a unique minimal point in $\langle \mathscr{Z} \rangle$.
- A *classical point* of a henselian rigid space $\mathscr{X}$ is a point-like locally closed rigid subspace $\mathscr{Z} \subseteq \mathscr{X}$.

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- $\mathscr{Z} \subseteq \mathscr{X}$ is in fact a closed subspace.
- $\mathscr{Z} = (\mathrm{Sph}\, W)^{\mathrm{rig}}$, where W is finite, flat, and finitely presented over V.

Classical points

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- A: henselian admissible V-algebra,
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Proposition

For any classical point $\mathscr{Z} \hookrightarrow \mathscr{X}$, the image of $s(\mathscr{Z}) \to s(\mathscr{X})$ is a closed point, and this establishes a canonical bijection

$$\left\{\begin{array}{l}\text{classical points}\\ \text{of } \mathscr{X}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{l}\text{closed points}\\ \text{of } s(\mathscr{X})\end{array}\right\}.$$

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- $\mathscr{X}^{\mathrm{cl}}$ = the set of all classical points of $\mathscr{X}$.
- $\mathscr{X} \mapsto \mathscr{X}^{\mathrm{cl}}$ is functorial.

Quasi-finite morphisms

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Definition

φ is said to be *quasi-finite*

$\overset{\text{def}}{\Longleftrightarrow}$ for any $x \in \mathscr{Y}^{\mathrm{cl}}$ the fiber $\mathscr{X} \times_{\mathscr{Y}} x$ is of dimension 0,

$\Leftrightarrow$ for any $x \in \mathscr{Y}^{\mathrm{cl}}$ the fiber $\mathscr{X} \times_{\mathscr{Y}} x$ consists of finitely many classical points.

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- If $f \colon X \to Y$ is a quasi-finite map between sep. finite type schemes over K, then $f^{\mathrm{an}} \colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is quasi-finite.

Contents

Statement

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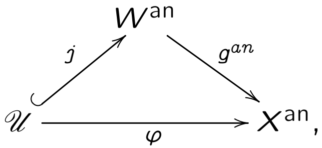
Theorem

Let X be a separated finite type scheme over K, $\mathscr{U} = (\mathrm{Sph}\, A)^{\mathrm{rig}}$ a henselian affinoid space of finite type over K, and $\varphi\colon \mathscr{U} \to X^{\mathrm{an}}$ a quasi-finite K-morphism. Then there exists a finite *morphism $g\colon W \to X$ with the commutative diagram*



where j is an open immersion.

Proof

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- Write $A = \varinjlim A_\lambda$ such that
  - each A_λ is finite type V-algebra;
  - $A_\lambda \to A_\mu$ is étale with $A_\lambda / aA_\lambda \cong A_\mu / aA_\mu$;
  - $A_\lambda^h = A$ for each λ.

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  - $A_\lambda \to A_\mu$ is étale with $A_\lambda/aA_\lambda \cong A_\mu/aA_\mu$;
  - $A_\lambda^h = A$ for each λ.
- The map $\varphi\colon \mathscr{U} = (\mathrm{Sph}\, A)^{\mathrm{rig}} \to X^{\mathrm{an}}$ gives $\tilde{\varphi}\colon\ \mathrm{Spec}\, A[\frac{1}{a}] \to X$.

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Proof

- Write $A = \varinjlim A_\lambda$ such that
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- Observe: $\tilde{\varphi}_\lambda$ is quasi-finite (due to Chevalley's Theorem).

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- $\mathscr{U} = (\operatorname{Sph} A)^{\mathrm{rig}} = (\operatorname{Sph} A^h_\lambda)^{\mathrm{rig}}$ is an affinoid domain in $(\operatorname{Spec} A_\lambda[\frac{1}{a}])^{\mathrm{an}}$. □

Scheme-theoretic closure

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Corollary

Let X be a separated finite type scheme over K, $\mathscr{U}$ a henselian rigid space of finite type over K, and $\varphi\colon \mathscr{U} \hookrightarrow X^{\mathrm{an}}$ an immersion. Then there exists a closed subscheme *$W \subseteq X$ that is smallest among those containing the image of $\mathscr{U}$ as an open subspace.*

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- Take $W \to X$ as in the theorem.
- Replace it by the scheme-theoretic image (in the usual sense) in X. □

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Proposition

For any flat family $\mathscr{Y} \subseteq X^{\mathrm{an}} \times_K \mathscr{U}$ over finite type henselian affinoid $\mathscr{U}$, there exists an affine finite type scheme U such that

(a) *U^{an} contains $\mathscr{U}$ as an affinoid subdomain;*

(b) *$\mathscr{Y}$ extends to a flat family $Y \subseteq X \times_K U$ over U.*

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- Observe: $Y \to U = \mathrm{Spec}\, A[\frac{1}{a}]$ is the projective limit of $Y_\lambda \to U_\lambda$.

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- Observe: $Y \to U = \mathrm{Spec}\, A[\frac{1}{a}]$ is the projective limit of $Y_\lambda \to U_\lambda$.
- Y is U-flat, since $\mathscr{Y}$ is $\mathscr{U}$-flat.
- By standard limit argument, there exists λ such that Y_λ is U_λ-flat, hence giving a desired extension. □