Metrics and Delta-Forms in Non-Archimedean Analytic Geometry

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Non-Archimedean Analytic Geometry: Theory and Practice
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Report on joint work with Walter Gubler (Regensburg)

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[GK] Walter Gubler and Klaus Künnemann. A tropical approach to non-archimedean Arakelov geometry.
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[Gu] Walter Gubler: Forms and currents on the analytification of an algebraic variety (after Chambert–Loir and Ducros). arXiv:1303.7364

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- Arithmetic intersection theory gives arithmetic information about X, e.g. *heights* $\rightsquigarrow$ finiteness results.
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- Algebraic intersection theory gives geometric information about X, e.g. degree of varieties.
- Arithmetic intersection theory gives arithmetic information about X, e.g. *heights* $\rightsquigarrow$ finiteness results.
  - $\dim X = 1$: Arakelov, Faltings
  - $\dim X \geq 2$: Gillet-Soulé,...
- Arithmetic intersection numbers can often be *localized* i.e. they are given as a sum of contributions from the *archimedean* and the *non-archimedean* places of K.

Basic idea of arithmetic intersection theory

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for a differential form $\omega_Z \in A^{p,p}(X(\mathbb{C}))$.

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- $f \in K(X)^\times$ gives arithmetic cycle $\widehat{div}(f) := (div(f), -\log|f|)$.

Arithmetic intersection theory

- *Arithmetic intersection product* for $(\mathfrak{Y}, g_Y)$ and $(\mathfrak{Z}, g_Z)$ such that Y intersects Z properly in X is given as

$$(\mathfrak{Z}, g_Z) \cdot (\mathfrak{Y}, g_Y) := (\mathfrak{Y} \cdot \mathfrak{Z}, \underbrace{g_Y \wedge \delta_Z + \omega_Y \wedge g_Z}_{=: g_Y * g_Z}).$$

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Aim of non-archimedean Arakelov theory

Use analytic spaces over K_v for non-archimedean v instead of models and a similar analytic theory of currents.

Some notions from tropical geometry

- Recall that an *(integral, $\mathbb{R}$-affine) polyhedron* Δ in $\mathbb{R}^r$ is a set

$$\Delta := \bigcap_{i=1}^{N} \left\{ \omega \in \mathbb{R}^r \,\middle|\, \langle u_i, \omega \rangle \geq \gamma_i \right\}, \quad u_i \in \mathbb{Z}^r, \gamma_i \in \mathbb{R}.$$

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- A *polyhedral complex* Σ in $\mathbb{R}^r$ consists of a finite set Σ of polyhedra in $\mathbb{R}^r$ such that
  - $\Delta \in \Sigma \;\Rightarrow$ every face of Δ is in Σ,
  - $\Delta, \Delta' \in \Sigma \;\Rightarrow \Delta \cap \Delta'$ is empty or a face of Δ and Δ'.

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- A *weighted polyhedral complex* (Σ, m) of pure dimension n consists of a polyhedral complex Σ of pure dimension n and a weight function

$$m : \Sigma_n := \{\Delta \in \Sigma | \dim(\Delta) = n\} \to \mathbb{Z}.$$

Lagerberg's superforms

- Following Lagerberg we define for $U \subseteq \mathbb{R}^n$ open

$$A^{p,q}(U) := A^p(U) \otimes_{C^\infty(U)} A^q(U) = A^p(U) \otimes_{\mathbb{Z}} \bigwedge^q \mathbb{Z}^r$$

$\rightsquigarrow$ bigraded differential alternating algebra $(A^{\cdot,\cdot}, d', d'')$.

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- In local coordinates and with multi-index notation

$$\begin{aligned}
\alpha &= \sum_{|I|=p,|J|=q} f_{IJ} d'x_I \wedge d''x_J, \\
d'\alpha &= \sum_{i=1}^{r} \sum_{I,J} \frac{\partial f_{IJ}}{\partial x_i} d'x_i \wedge d'x_I \wedge d''x_J, \\
d''\alpha &= \sum_{j=1}^{r} \sum_{I,J} \frac{\partial f_{IJ}}{\partial x_j} d''x_j \wedge d'x_I \wedge d''x_J.
\end{aligned}$$

Tropical geometry and superforms

- Integration of $\alpha \in A_c^{n,n}(\mathbb{R}^r)$ over n-dimensional polyhedron Δ is well defined
  $\rightsquigarrow$ current $\delta_\Delta \in D_{n,n}(\mathbb{R}^r) =: D^{r-n,r-n}(\mathbb{R}^r)$

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- For a weighted polyhedral complex (Σ, m) of pure dimension n
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- (Mikhalkin, Allermann, Rau, ...) There is a well-defined intersection product for tropical cycles on $\mathbb{R}^r$ (no equivalence relation is needed).

δ-preforms

Definition

A current in $D^{p,q}(\mathbb{R}^r)$ is a δ*-preform* of type (p, q) if and only if it is of the form

$$\sum_{i=1}^{N} \alpha_i \wedge \delta_{C_i}$$

with $\alpha_i \in A^{p_i,q_i}(\mathbb{R}^r)$ and $C_i = (\Sigma_i, m_i)$ tropical cycles of codimension k_i with $(p, q) = (p_i + k_i, q_i + k_i)$

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Get bigraded differential algebra $(P^{\bullet,\bullet}(\mathbb{R}^r), d', d'')$, where

$$(\alpha_i \wedge \delta_{C_i}) \wedge (\alpha'_j \wedge \delta_{C'_j}) := (\alpha_i \wedge \alpha'_j) \wedge \delta_{C_i \cdot C'_j}$$

using the tropical intersection product.

Non-archimedean analytification

- From now on, K denotes an algebraically closed field complete with respect to a non-trivial non-archimedean absolute value $| \ | : K \to [0, \infty)$ with valuation ring K° and residue class field $\tilde{K}$.

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- For closed subvariety U of a torus $T = \mathbb{G}_m^r$ over K, consider

$$\mathrm{trop} : T^{\mathrm{an}} \longrightarrow \mathbb{R}^r,\ p \longmapsto (-\log p(t_1), \ldots, -\log p(t_r))$$

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Theorem (Bieri, Groves, Speyer, Sturmfels)

$\mathrm{Trop}(U)$ *has a canonical structure of a tropical cycle.*

Very affine open subsets

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- Choosing a $\mathbb{Z}$-basis $f_1, \ldots, f_r$ of M_U, we obtain (up to translation in T) a canonical map

$$\varphi_U = (f_1, \ldots, f_r) : U \to \mathbb{G}_m^r = T_U = \mathrm{Spec}\, K[M_U].$$

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- Call U *very affine*, if it satisfies the equivalent conditions
  - U admits closed embedding into a torus,
  - $\mathcal{O}(U)$ is generated as a K-algebra by $\mathcal{O}(U)^\times$,
  - φ_U is a closed embedding.

Tropical charts

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Definition

A *tropical chart* (V, φ_U) on X consists of

- a very affine open subset U of X
- an associated closed immersion $\varphi_U : U \hookrightarrow T_U = \mathbb{G}_m^r$,
- an open subset Ω of $\mathrm{Trop}(U)$ such that V is the open subset $\mathrm{trop}_U^{-1}(\Omega)$ in U^{an} where

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- Tropical charts form a basis for the topology on X^{an}.
- Be careful: A tropical chart (V, φ_U) will not give complete local 'information' about V!

δ-forms and δ-currents

Definition

A δ*-form* α on X^{an} is given by tropical charts $(V_i, \varphi_{U_i})_{i \in I}$ covering X^{an} and a family $\alpha = (\alpha)_{i \in I}$ where $\alpha_i \in P^{\bullet,\bullet}(N_{U_i,\mathbb{R}})$ such that

$$\alpha = \alpha' \Leftrightarrow \alpha_i\big|_{V_i \cap V_j'} = \alpha_j'\big|_{V_i \cap V_j'} \quad \text{in a tropical sense (see [GK]).}$$

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- Topological dual $B_c^{\bullet,\bullet}(X^{\mathrm{an}})$ is space of *δ-currents* $E^{\bullet,\bullet}(X^{\mathrm{an}})$.
- Observe that no smoothness assumption on X is required.
- Consider δ-forms as analogs of complex differential forms with logarithmic singularities.

Smooth forms after Chambert-Loir, Ducros

- Using Lagerberg's superforms and skipping tropical cycles $\overset{\textit{similarly}}{\rightsquigarrow}$ smooth (p, q)-forms of Chambert-Loir and Ducros (see [CD], [Gu]) leading to subalgebra $A^{\bullet,\bullet}(X^{\mathrm{an}})$ of $B^{\bullet,\bullet}(X^{\mathrm{an}})$.

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- Chambert-Loir, Ducros work more generally on analytic spaces with boundary (no balancing condition is known at the boundary).
- Given a tropical chart (V,φ_U), $f_1,\ldots,f_m \in \mathcal{O}(U)^\times$ and $g : \mathbb{R}^m \to \mathbb{R}$ smooth, the function

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is smooth. Not all functions in $A^0(V)$ are of this form!

Smooth forms after Chambert-Loir, Ducros

- Using Lagerberg's superforms and skipping tropical cycles $\overset{similarly}{\rightsquigarrow}$ smooth (p, q)-forms of Chambert-Loir and Ducros (see [CD], [Gu]) leading to subalgebra $A^{\bullet,\bullet}(X^{\mathrm{an}})$ of $B^{\bullet,\bullet}(X^{\mathrm{an}})$.
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- Cohomology of $(A^{0,\cdot}(X^{\mathrm{an}}), d'')$ computes singular cohomology of X^{an} (Philipp Jell, `arXiv:1409.0676`).

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- Get $C_c^0(W) \to E^{0,0}(W),\ f \mapsto [f]$.

Metrics on line bundles

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- Let $W \subseteq X^{\mathrm{an}}$ be open. Given sections $s \in \Gamma(W, L^{\mathrm{an}})$, the metric $\| \ \|$ determines continuous functions

$$\|s\| : X^{\mathrm{an}} \longrightarrow \mathbb{R}$$

  such that $\|s_i\| = \rho_i$.

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Similarly as in [CD], we get

Poincaré-Lelong formula

$$d'd''[-\log \|s\|] + \delta_{div(s)} = [c_1(L, \| \quad \|)] \quad \text{in } E^{1,1}(X^{\mathrm{an}})$$

for any meromorphic section s of L over X.

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Problem: Algebraic and formal metrics are not always smooth!

Delta-metrics

Definition

A δ-*metric* on L is a piecewise smooth metric $\| \quad \|$ such that the δ-current $[c_1(L, \| \quad \|)]$ is represented in an functorial way by a δ-form $c_1(L, \| \quad \|)$.

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- Canonical metrics on line bundles algebraically equivalent to zero are δ-metrics.

Measures associated with metrized line bundles

- **Chambert-Loir measure:** Let X be projective and $\| \ \|_{\mathcal{L}}$ an algebraic metric on L^{an} given by $(\mathfrak{X}, \mathcal{L})$ such that the special fibre $\mathfrak{X}_s$ is reduced.

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$$\mu = \sum_Y deg_{\mathcal{L}}(Y)\delta_{\xi_Y}$$

where Y ranges over the irreducible components of $\mathfrak{X}_s$ and ξ_Y is the unique point of X^{an} whose reduction is the generic point of Y.

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  where Y ranges over the irreducible components of $\mathfrak{X}_s$ and ξ_Y is the unique point of X^{an} whose reduction is the generic point of Y.
- **Monge-Ampère measure:** Let $\| \ \|$ be a δ-metric on L^{an}, i.e. $c_1(L, \| \ \|)$ is a δ-form. Then $c_1(L, \| \ \|)^{\wedge n} \in B^{n,n}(X^{an})$ and it induces a measure on X^{an}.

Chambert-Loir measure = Monge-Ampère measure

Let $(L, \| \ \|)$ be an algebraic metric.

Theorem (GK)

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A variant of this following Theorem was proved before by Chambert-Loir and Ducros.

Difference to [CD]:

- Chambert-Loir and Ducros use an approximation process by smooth metrics as in Bedford-Taylor theory to define $[c_1(L, \|\ \ \|)]^{\wedge n}$ as a wedge product of currents.
- We obtain $c_1(L, \|\ \ \|)^{\wedge n}$ *directly* as a wedge product of δ-forms. This simplifies the proof.

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holds for some δ-form $\omega(Z, g_Z) := \omega_Z \in B^{p,p}(X^{\mathrm{an}})$.

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- Poincaré-Lelong equation $\Rightarrow g_D = [-\log \|s\|] \in E^{1,1}(X^{\mathrm{an}})$ is a Green current for D with $\omega_D = c_1(L, \| \ \|)$.

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If D intersects Z property, then

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- The $*$-product is commutative modulo $\mathrm{Im}\,(d') + \mathrm{Im}\,(d'')$.

Local heights

Definition: Let $D_0, \ldots, D_n$ be Cartier divisors intersecting properly on X. Let $\mathcal{O}(D_0), \ldots, \mathcal{O}(D_n)$ be equipped with δ-metrics $\| \ \|_i$ and associated Green currents $g_{D_i} = [-\log \|s_{D_i}\|_i]$ then

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Theorem (GK)

If we use algebraic metrics on $\mathcal{O}(D_0), \dots, \mathcal{O}(D_n)$, then $\lambda(X)$ is the usual local height of X in arithmetic geometry given as the intersection number of the Cartier divisors on a corresponding model.

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- Call $\| \ \|$ *semipositive* on W if it is semipositive at $x \ \forall x \in W$.

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(v) The restriction of $\| \ \|$ to $W \cap C^{\mathrm{an}}$ is psh for any closed curve C of X.

The proof is based on a lifting theorem which allows to lift curves from the special to the generic fibre of an algebraic model.

Piecewise smooth psh metrics

Corollary: Let the metric $\| \ \|$ on L^{an} be uniformly approximable by semipositive formal metrics on L^{an}.

(i) If $\| \ \|$ is formal then it is semipositive.

(ii) Pullback of $\| \ \|$ to any (not necessarily proper) curve is psh.

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(i) the restriction of ϕ to each maximal face of $\mathrm{Trop}(U)$, where ϕ is smooth, is a convex function and

(ii) the corner locus $\phi \cdot \mathrm{Trop}(U)$ is effective tropical cycle.

Thank you for your attention!