Closed Points on Torsors under Abelian Varieties

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$\overline{A}^0$: the identity component of the special fiber of A

Definition A_K has an (*ordinary*) *semi-abelian reduction* $\overline{A}^0$
if $\overline{A}^0$ is an extension of an (ordinary) $\overline{K}$-abelian variety by a $\overline{K}$-torus

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Definition A_K has an (*ordinary*) *semi-abelian reduction* $\overline{A}^0$
if $\overline{A}^0$ is an extension of an (ordinary) $\overline{K}$-abelian variety by a $\overline{K}$-torus

X_K : a K-torsor under A_K corresponding to $a \in H^1(K, A_K)$

$$H^1(K, A_K) := H^1(\mathrm{Gal}(K^{\mathrm{sep}}/K), A_K(K^{\mathrm{sep}})) \xleftrightarrow{1:1} \{\text{isomorphism classes of torsors under } A_K\}$$

$$(A_K \times_K X_K \longrightarrow X_K \times_K X_K,\ (g, x) \longmapsto (gx, x) \text{ is a } K\text{-isomorphism})$$

$P(X_K) := P(a) :=$ the order of a : the *period* of X_K

1. Main Theorem

Notation

- $\hat{\mathbf{Z}}$: the profinite completion of the integers
- $[k' : k]$: the degree of a field extension k'/k
- $k(x)$: the residue field of a scheme at a point x

Definition Let k be a field

- k is *quasi-finite* (resp. *weakly quasi-finite*) if $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ is isomorphic to $\hat{\mathbf{Z}}$ (resp. a quotient of $\hat{\mathbf{Z}}$)
- k is *WC-trivial* if $H^1(k, B_k)=0$ for any k-abelian variety B_k
- A k-torus T_k of dim t *splits over* k if T_k is k-isomorphic to $(\mathbf{G}_{m,k})^t$ as k-group schemes
- A point x on a k-scheme is *separable* if $k(x)/k$ is separable

1. Main Theorem

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A_K : a K-abelian variety of dim g

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Main Theorem

Assume (a) $\overline{K}$ is perfect, weakly quasi-finite, and WC-trivial,
and (b) A_K has an ordinary semi-abelian reduction $\overline{A}^0$

$\overline{T}$: the torus part of $\overline{A}^0$

Suppose (1) $\overline{T}$ splits over $\overline{K}$, or (2) dim $\overline{T} \leq \max\{g - 2, 0\}$

Then $\exists$ a separable closed point x on X_K such that $[k(x) : K] \mid P(X_K)^g$

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Remark

Condition (a) is satisfied whenever $\overline{K}$ is algebraically closed or finite

Condition (1) is satisfied whenever $\overline{K}$ is algebraically closed or $\overline{T} = 0$

2. Motivations and Conclusions

Closed point of small degree [Gerritzen 73]

Z_K : a K-scheme of finite type $\longmapsto$ Z_K^{an} : the analytification of Z_K

$u_K : E_K^{\mathrm{an}} \longrightarrow A_K^{\mathrm{an}} = E_K^{\mathrm{an}}/\Gamma_K$: a rigid analytic uniformization of A_K,
where E_K is a K-semi-abelian variety

For a general cdvf K, if E_K is a split K-torus and Γ_K is a split K-lattice,
then the conclusion of Main Theorem holds

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Remark [Lang–Tate 58]

$\exists$ a cdvf K with perfect quasi-finite residue field (non-WC-trivial),
$\exists$ a K-elliptic curve A_K with ordinary good reduction,
and $\exists$ a non-trivial K-torsor X_K under A_K
such that $\forall$ separable closed point x on X_K, $P(X_K)^2 \mid [k(x) : K]$

2. Motivations and Conclusions

Index (Period–index problem)

$\mathrm{Br}(K) :=$ {Brauer classes of central simple K-algebras}

$\overset{1:1}{\longleftrightarrow}$ $H^2(K, \mathbf{G}_{m,K}) := H^2(\mathrm{Gal}(K^{sep}/K), \mathbf{G}_{m,K}(K^{sep}))$

{isomorphism classes of torsors under A_K}

$\overset{1:1}{\longleftrightarrow}$ $H^1(K, A_K) := H^1(\mathrm{Gal}(K^{sep}/K), A_K(K^{sep}))$

$I(X_K) := \gcd\{[k(x) : K] \mid x$ is a closed point on $X_K\}$: the *index* of X_K

Then $I(X_K) =$ the minimum positive integer

among the degrees of all zero-cycles on X_K

$I(X_K) \mid P(X_K)^{2g}$ for any field K and any K-abelian variety A_K

$I(X_K) \mid P(X_K)^{g}$ in the case of Main Theorem

2. Motivations and Conclusions

The exponent g is best possible [Gerritzen 73]: $\forall\ g > 0$, $\forall\ n > 0$,
$\exists$ a local field K, $\exists$ a K-abelian variety A_K of dim g with split toric reduction, $\exists$ a K-torsor X_K under A_K such that
$P(X_K) = n$ and $I(X_K) = P(X_K)^g$

Global field K [Sharif 12]: $\forall$ K-elliptic curve A_K, $\forall$ n > 0,
$\exists$ a K-torsor X_K under A_K such that $P(X_K) = n$ and $I(X_K) = P(X_K)^2$

Local field K [Clark 10]: $\forall\ g > 0$, $\exists$ c > 0 such that
$\forall$ *principally polarized* K-abelian variety A_K of dim g,
$\forall$ K-torsor X_K under A_K, $I(X_K) \leq cP(X_K)^g$
[Lichtenbaum 68] and [Milne 70]: if $g = 1$, then $I(X_K) = P(X_K)$

3. Proof of Main Theorem

E_K^{an} : a universal covering of $A_K^{\mathrm{an}} = E_K^{\mathrm{an}}/\Gamma_K^{\mathrm{an}}$

E : the identity component of a Néron model of E_K

If A_K has a semi-abelian reduction, then E is an $\mathcal{O}_K$-semi-abelian variety

$\mathcal{E}$: the formal completion of E along the special fiber

$\mathcal{E}^{\mathrm{rig}}$: the rigidification of $\mathcal{E}$

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$$H^1(K, E_K) \longrightarrow H^1(K, A_K) \longrightarrow H^2(K, \Gamma_K)$$

$$\alpha \longmapsto \gamma$$

(1) $\exists$ a finite separable extension K'/K, $\gamma|_{K'} = 0$ and $[K' : K] \mid P(\gamma)^t$

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$$H^1(K, E_K) \longrightarrow H^1(K, A_K) \longrightarrow H^2(K, \Gamma_K)$$

$$\beta \longmapsto \alpha \longmapsto 0 \qquad P(\beta) = P(\alpha)$$

(1) $\exists$ a finite separable extension K'/K, $\gamma|_{K'} = 0$ and $[K' : K] \mid P(\gamma)^t$

(2) $\exists$ a fintie separable extension K'/K, $\beta|_{K'} = 0$ and $[K' : K] \mid P(\beta)^{g-t}$

3. Proof of Main Theorem

Sheaves of modules on the small étale site of K

$$0 \longrightarrow \mathscr{E}^{\mathrm{rig}} \longrightarrow E_K^{\mathrm{an}} \longrightarrow \Phi_E \longrightarrow 0 \qquad \text{exact sequence}$$

unit group value group

Example $\Phi_{\mathbf{G}_{\mathrm{m}}}$: the constant sheaf associated with $|(K^{\mathrm{sep}})^{\times}| = \mathbf{Q}$

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Sheaves of modules on the small étale site of K

$$0 \longrightarrow \mathcal{E}^{\mathrm{rig}} \longrightarrow E_K^{\mathrm{an}} \longrightarrow \Phi_E \longrightarrow 0$$ exact sequence

unit group value group

Example $\Phi_{\mathbf{G}_{\mathrm{m}}}$: the constant sheaf associated with $|(K^{\mathrm{sep}})^{\times}| = \mathbf{Q}$

$$H^1(K_{\mathrm{\acute{e}t}}, \mathcal{E}^{\mathrm{rig}}) \longrightarrow H^1(K_{\mathrm{\acute{e}t}}, E_K^{\mathrm{an}}) \longrightarrow H^1(K_{\mathrm{\acute{e}t}}, \Phi_E) = 0 \qquad H^1(K, E_K) = H^1(K_{\mathrm{\acute{e}t}}, E_K^{\mathrm{an}})$$

$$\delta \longmapsto \beta \qquad\qquad \beta$$

$H^1(K_{\mathrm{\acute{e}t}}, \mathcal{E}^{\mathrm{rig}}) = H^1(\mathrm{Gal}(K^{\mathrm{sep}}/K), E(\mathcal{O}_{K^{\mathrm{sep}}}))$: models of torsors under E_K

Studies on their special fibers and the formal group law induced by E

4. Models of Torsors

S : an integral scheme
G : an fppf S-group scheme
X_η : an η-torsor under G_η

η : the generic point of S
G_η : the generic fiber of G
ρ_{G_η} : the η-action of G_η on X_η

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Renée Lewin–Ménégaux, Modèles minimaux de torseurs, présentée par Jean–Pierre Serre, C. R. Acad. Sci. Paris Sér. I Math. 297 (1983), no. 4, 257–260

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Definition

A *minimal S-model of X_η* is an S-model X of X_η locally of finite presentation with S-action ρ_G on X of G such that

(a) ρ_G is an extension of ρ_{G_η}

(b) $\exists$ a finite separable covering $\eta' \longrightarrow \eta$,
$\exists$ a trivialization $\tau : G_{\eta'} \longrightarrow X_{\eta'}$ of X_η

(c) S' : the normalization of S in η'
$\exists$ a G-equivariant integral faithfully flat S-morphism
$G \times_S S' \longrightarrow X$ that is an extension of $G_{\eta'} \xrightarrow[\tau]{} X_{\eta'} \xrightarrow[pr]{} X$

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S'/S : a finite Galois covering with Galois group $\mathrm{Gal}(S'/S)$

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$\alpha \in H^1(K, G_\eta)$

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Definition

α is said to *admit a model on G* (*over* S') if α is represented by an element of the image of the specialization map

$$Z^1(\mathrm{Gal}(S'/S), G(S')) \longrightarrow Z^1(\mathrm{Gal}(S'/S), G_\eta(\eta'))$$

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Theorem

Assume (1) S is a Dedekind scheme,

and (2) G is smooth and quasi-projective over S

Then $\exists$ a minimal S-model of X_η $\Leftrightarrow$ $\exists$ α admits a model on G

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Proof

Construction of a model $G \times_S S'/\mathrm{Gal}(S'/S)$