Theorems A and B for Berkovich spaces over $\mathbf{Z}$

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Outline

The analytic space $\mathbf{A}_A^{n,\mathrm{an}}$

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$$|.| : A[T_1, \dots, T_n] \to \mathbf{R}_+$$

such that

1. $|0| = 0$ and $|1| = 1$;
2. $\forall f, g \in A[T_1, \dots, T_n]$, $|f + g| \leq |f| + |g|$;
3. $\forall f, g \in A[T_1, \dots, T_n]$, $|fg| = |f|\,|g|$;
4. $\forall f \in A$, $|f| \leq \|f\|$.

The topology on $\mathbf{A}_A^{n,\mathrm{an}}$

The topology on $\mathbf{A}_A^{n,\mathrm{an}}$ is the coarsest topology such that, for any f in $A[T_1, \ldots, T_n]$, the evaluation function

$$\begin{array}{ccc} \mathbf{A}_A^{n,\mathrm{an}} & \to & \mathbf{R}_+ \\ |.|_x & \mapsto & |f|_x \end{array}$$

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Theorem (Berkovich)

The space $\mathbf{A}_A^{n,\mathrm{an}}$ is Hausdorff and locally compact.

The structure sheaf on $\mathbf{A}_A^{n,\mathrm{an}}$

Definition (Berkovich)

For every open subset U of $\mathbf{A}_A^{n,\mathrm{an}}$, $\mathscr{O}(U)$ is the set of maps

$$f\colon U \to \bigsqcup_{x\in U} \mathscr{H}(x)$$

such that

1. $\forall x \in U,\ f(x) \in \mathscr{H}(x)$;
2. f is locally a uniform limit of rational functions without poles.

Outline

Local properties of $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$

Theorem (Lemanissier)

The space $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$ is locally arcwise connected.

Theorem (P.)

- *For every x in $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$, the local ring $\mathscr{O}_x$ is henselian, noetherian, regular, excellent.*
- *The structure sheaf of $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$ is coherent.*

Outline

Closed disks

Let $(A, \|\cdot\|)$ be a Banach ring. Let $r_1, \ldots, r_n > 0$. Set

$$\overline{\mathbf{D}}_A(r_1, \ldots, r_n) = \{x \in \mathbf{A}_A^{n,\mathrm{an}} \mid \forall i, |T_i(x)| \leq r_i\}.$$

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We have

$$\begin{aligned}\mathscr{O}(\overline{\mathbf{D}}) &= \varinjlim_{U \supset \overline{\mathbf{D}}} \mathscr{O}(U) \\ &= \varinjlim_{s_i > r_i} A\langle s_1^{-1} T_1, \ldots, s_n^{-1} T_n \rangle,\end{aligned}$$

where

$$A\langle s_1^{-1} T_1, \ldots, s_n^{-1} T_n \rangle = \Big\{ \sum_{u \ge 0} b_u T^u \mid \sum_{u \ge 0} \|b_u\| r^u < \infty \Big\}.$$

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There exists $C > 0$ such that, for every $s \in \mathscr{O}(L)$, there exist $s^\pm \in \mathscr{O}(K^\pm)$ such that

1. $s = s^- + s^+$;
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In fact, we need more:

- $\varinjlim \mathscr{C}_k \xrightarrow{\sim} \mathscr{O}(L)$
- $\varinjlim \mathscr{B}_k^\pm \to \mathscr{O}(K^\pm)$
- $\forall k,\ \mathscr{B}_k^\pm \to \mathscr{C}_k$
- $s \in \mathscr{C}_k,\ s^\pm \in \mathscr{B}_k^\pm$

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Runge property

For all finite families $(s_i), (t_j)$ in $\mathscr{O}(L)$, there exist f in $\mathscr{O}(K^+)$ invertible and families (s_i') in $\mathscr{O}(K^-)$ and (t_j') in $\mathscr{O}(K^+)$ that approximate $(f^{-1}s_i)$ and $(f\,t_j)$ arbitrarily well.

Cousin-Runge systems II

Let (K^-, K^+) be a Cousin-Runge system. Let $\mathscr{F}$ be a sheaf of finite type on $M = K^- \cup K^+$.

Proposition

If $\mathscr{F}$ is generated by global sections on $K^{\pm}$,

i.e. for every x in $K^{\pm}$, $\mathscr{F}_x$ is generated by $\mathscr{F}(K^{\pm})$ as an $\mathscr{O}_x$-module,

then $\mathscr{F}$ is generated by global sections on M.

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Every sheaf of finite type on $\overline{\mathbf{D}}$ is generated by global sections.

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Theorem (Theorem A)

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Theorem (Theorem B)

For every coherent sheaf $\mathscr{F}$ on $\overline{\mathbf{D}}$ and every $q \geq 1$, we have

$$H^q(\overline{\mathbf{D}}, \mathscr{F}) = 0.$$

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Affinoid spaces over $\mathbf{Z}$

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An affinoid space X is of the form $(V(\mathscr{I}), \mathscr{O}/\mathscr{I})$, where $\mathscr{I}$ is a coherent sheaf on some $\overline{\mathbf{D}} = \overline{\mathbf{D}}_{\mathbf{Z}}(r_1, \ldots, r_n)$.

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Theorems A and B hold for affinoid spaces.

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Theorem

If $\mathscr{I} = (f_1, \dots, f_m)$, *then*

$$\mathscr{O}(X) \simeq \mathscr{O}(\overline{\mathbf{D}})/(f_1, \dots, f_m).$$