Skeleton and Dual Complex

Chenyang Xu

Beijing International Center of Mathematics Research

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- Let $K = k((t))$, where $\mathrm{char}(k) = 0$ and $R = k[[t]]$. We fix a t-adic absolute value on K by setting $|t|_K = 1/e$.
- Let X be a smooth proper variety over K and $\mathcal{X}$ a proper model over R. Denote by X_k special fiber and X_k^{red} the reduced special fiber. We assume $\mathcal{X}$ is obtained from a base change of an algebraic model over $\mathcal{O}_{p,C}$, where C is a curve over k and $p \in C$ is a k point.
- Let X^{an} be the analytification defined by Berkovich.

- Assume $(\mathcal{X}, X_k^{\mathrm{red}})$ is simple normal crossing. Let $X_k^{\mathrm{red}} = \sum E_i$.
- Then we can define the dual complex $\mathcal{D}(X_k^{\mathrm{red}})$ in the following way: For each component E_i of E, we put a vertex v_{E_i}; for each irreducible component of $v_{E_i} \cap v_{E_j}$, we associate an edge connecting v_{E_i} and v_{E_j}; and for each irreducible component of $v_{E_i} \cap v_{E_j} \cap v_{E_k}$, we associate a 2-dimensional face, etc.. Eventually, we obtain a cell complex.
- Properties needed on $\sum E_i$: each component of the intersections has the expected dimension; the irreducible components of $\bigcap E_i$ coincide with the connected components.

Theorem (Berkovich, Thuillier)

If $(\mathcal{X}, X_k^{\mathrm{red}})$ is simple normal crossing, X^{an} has a strong deformation $\rho_{\mathcal{X}}\colon X^{\mathrm{an}} \to \mathrm{Sk}(\mathcal{X}) \simeq \mathcal{D}(X_k^{\mathrm{red}})$.

Corollary (Arapura-Bakhtary-Wlodarczyk, Payne, Stepanov, Thuillier)

$D(X_{1,k}^{\mathrm{red}})$ and $D(X_{2,k}^{\mathrm{red}})$ are (simple)-homotopical equivalent to each other if $\mathcal{X}_i$ are two snc models.

The corollary can be proved by using

Theorem (Abramovich-Karu-Matsuki-Wlodarczyk)

If $f_i: (\mathcal{X}_i, X_{i,k})$ are two snc birational models of X. Then there exists a sequence of admissible blow ups

$$\mathcal{X}_1 = \mathcal{Y}_1 \dashrightarrow \mathcal{Y}_2 \dashrightarrow \cdots \dashrightarrow \mathcal{Y}_n = \mathcal{X}_2,$$

such that $\mathcal{Y}_i \dashrightarrow \mathcal{Y}_{i+1}$ is an admissible blow up or its inverse.

- Fix (X, ω) where ω is a m-th pluri-canonical form, Kontsevich-Soibelman (2006) and Mustaţă-Nicaise (2012) define some interesting (sub)-skeleta via a weight function.
- Let $x \in X^{\mathrm{an}}$ be a monomial point, the weight function $\mathrm{wt}_\omega(x) = v_x(\mathrm{div}_{\mathcal{X}}(\omega) + m(X_k^{\mathrm{red}}))$.
- If E_i is a divisor on the special fiber with multiplicity N_i in X_k, and let $\mu_i = m + \mathrm{order}_\omega(E_i)$, then $\mathrm{wt}_\omega(x_{E_i}) = \mu_i / N_i$.

- wt_ω can be naturally extended as a lower semi-continuous function on X^{an}, by

$$\mathrm{wt}_\omega(x) = \mathrm{Sup}_{\mathcal{X}}\{\mathrm{wt}_\omega(\rho_{\mathcal{X}}(x))\} \in \mathbb{R} \cup \{+\infty\}.$$

- We define the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X,\omega) = \{p \in X^{\mathrm{an}} |\ \mathrm{wt}_\omega(X)(p)$ takes the minimum$\}$ and the essential skeleton $\mathrm{Sk}(X) = \bigcup_\omega \mathrm{Sk}(X,\omega)$ where ω runs over all pluri-canonical forms.
- $\mathrm{Sk}(X,\omega) \subset \mathrm{Sk}(\mathcal{X})$ for any snc model $\mathcal{X}$.

- Assume that $|\omega_X^{\otimes m}|$ is base point free for some $m \in \mathbb{Z}_{>0}$, i.e., X is a good minimal model. For example X is a CY manifold, i.e., $\omega_X \sim \mathcal{O}_X$.
- Applying the minimal model program, we can construct a minimal model $\mathcal{X}^{\min}$ over R such that $m(K_{\mathcal{X}^{\min}} + X_k^{\min,\mathrm{red}})$ is base point free over R for some $m \in \mathbb{Z}_{>0}$.
- Caveat: $(\mathcal{X}^{\min}, X_k^{\min,\mathrm{red}})$ has **divisorial log terminal** (dlt) singularities. Nevertheless, we can define $\mathcal{D}(X_k^{\min,\mathrm{red}})$.

Theorem (Nicaise-X. 2013)

$$\mathcal{D}(X_k^{\mathrm{min,red}}) \simeq \mathrm{Sk}(X).$$

- $\mathcal{D}(X_k^{\mathrm{min,red}}) \subset \mathrm{Sk}(X)$ is easy.
- For other side, assume $x \in \mathrm{Sk}(X, \omega)$. If $\mathrm{red}(x) \in \mathcal{X}^{\mathrm{min,snc}}$, then $x \in D(X_k^{\mathrm{min,red}})$.
- If $\mathrm{red}(x)$ is not in $\mathcal{X}^{\mathrm{min,snc}}$. Let θ be a generator of $\omega^{\otimes m}(mX_k)$ and write $\omega = g \cdot \theta$.
- Then $\mathrm{wt}_\omega(x) > \mathrm{In}|g(x)| \geq -\mathrm{In}|g(x')| = \mathrm{wt}_\omega(x')$ for some divisorial point x' corresponding to a component of $X_k^{\mathrm{min,red}}$.

Corollary

$D(X_{1,k}^{\mathrm{min,red}})$ and $D(X_{2,k}^{\mathrm{min,red}})$ are homeomorphism to each other if $\mathcal{X}_i^{\mathrm{red}}$ are two minimal models.

This can be proved by weak factorization, dlt properties and the fact that the pull back of $K_{\mathcal{X}_i^{\mathrm{min}}} + X_{i,k}^{\mathrm{min,red}}$ on a common resolution are the same.

Theorem (de Fernex-Kollár-X. 2012)

$\mathcal{D}(X_k^{\mathrm{red}})$ collapses to $\mathcal{D}(X_k^{\mathrm{min,red}})$.

The Theorem is proved by tracking how dual complexes vary during the minimal model program.

Corollary

X^{an} has a strong deformation retract to $\mathrm{Sk}(X)$.

Theorem (de Fernex-Kollár-X. 2012)

If X is a rationally connected varieties, then $\mathrm{Sk}(\mathcal{X})$ is contractible to a point for any snc model $\mathcal{X}$.

- Brown-Foster generalizes this result to a relative setting.

Theorem (Kollár-Kovács 2009)

The essential skeleton of a Calabi-Yau is a pseudo-manifold with boundary.

Question

Is the essential skeleton of a Calabi-Yau always a finite quotient of a sphere?

Question (Kontsevich-Soibelman)

If $\mathcal{X}$ is a simply connected CY admitted a minimal semi-stable degeneration. Assume $\dim \mathcal{D}(X_k) = \dim(X) - 1$. Is $\mathcal{D}(X_k)$ a PL-sphere?

- Consider a projective dlt pair (D, E) such that $K_D + E \sim_{\mathbb{Q}} 0$, we call it **a log Calabi-Yau (CY)**.
- Example: Let $(\mathcal{X}, X_0^{\mathrm{red}})$ be a minimal degeneration of Calabi-Yau, i.e. $(\mathcal{X}, \mathrm{red}(X_0))$ is dlt and $K_{\mathcal{X}} + X_0^{\mathrm{red}}) \sim_{\mathbb{Q}} 0$. If D is a component of X_0, then (D, E) is a log CY, where $(K_{\mathcal{X}} + X_0^{\mathrm{red}})\,|_D = K_D + E$.
- $\mathcal{D}(E)$ is the link of $\mathcal{D}(X_0^{\mathrm{red}})$ at v_E.

Question

Is $\mathcal{D}(E)$ a finite quotient of a sphere?

Theorem (Kollár-X. 2015)

Let (D, E) *be a log CY. Assume* $\dim(\mathcal{D}(E)) \geq 2$. *Then*

1. $H^i(\mathcal{D}(E), \mathbb{Q}) = 0$ *for* $1 \leq i \leq \dim\ \mathcal{D}(E) - 1$.
2. *There is a surjection* $\pi_1(D^{\mathrm{sm}}) \to \pi_1(\mathcal{D}(E))$.
3. *The profinite completion is* $\hat{\pi}_1(\mathcal{D}(E))$ *finite.*
4. *the finite cover of* $\mathcal{D}(E)$ *given by* $\hat{\pi}_1(\mathcal{D}(E))$ *is the dual complex of a log CY.*

We will only discuss the case that (D, E) is snc and $\dim(D(E)) = \dim(D) - 1$. Then (2)-(4) just says that $\pi_1(D) = \pi_1(\mathcal{D}(E)) = \{e\}$.

Vanishing of rational cohomology

- There is an injection $H^i(\mathcal{D}(E), \mathbb{C}) \to H^i(E, \mathcal{O}_E)$.
- We have the exact sequence

$$H^i(\mathcal{O}_X(-E)) \to H^i(\mathcal{O}_X) \to H^i(\mathcal{O}_E),$$

  and $H^i(\mathcal{O}_X(-E)) \cong H^{\dim X - i}(\mathcal{O}_X)$,
- when (X, E) is a log CY, the first two terms vanish.

Change the models

Theorem (Maximal Boundary Theorem, Kollár-X)

There is a birational model (G, Δ) *of* (D, E) *such that*

1. (G, Δ) *is log canonical,* (D, E) *and* (G, Δ) *are crepant birationally equivalent.*
2. Δ *supports a divisor* H *which is ample over a variety* Z *such that* $\dim Z \leq \dim(D) - \dim(\mathcal{D}(E)) - 1$.
3. *Furthermore,* $D \dashrightarrow G$ *is isomorphic over* $G \setminus \Delta$.

- Assume (G, Δ) to be snc (though this is a very restrictive assumption).
- We have $\{e\} = \pi_1(G) \cong \pi_1(\Delta) \twoheadrightarrow \pi_1(\mathcal{D}(\Delta))$ and $\mathcal{D}(\Delta) \cong \mathcal{D}(E)$.
- The first isomorphism comes from $\pi_1(G) = \pi_1(D)$. The second isomorphism comes from Lefschetz hyperplane theorem.
- In general, we have to understand the difference between $D^{\rm sm}$ and $G^{\rm sm}$ and apply singular Lefschetz hyperplane theorem as in Goresky-MacPherson's book.

Question

If (D, E) has a maximal boundary, i.e., (D, E) is snc log CY, is $H^i(\mathcal{D}(E), \mathbb{Z}) = 0$ for $1 \leq i \leq \dim\ \mathcal{D}(E) - 1$?

- If this is true, by inductively applying Poincaré conjecture, we know that $\mathcal{D}(E)$ is a topological sphere.
- Our theorem implies $\mathcal{D}(E)$ is a topological sphere for dimension at most four.
- So if X is a simply connected CY manifold with $\dim(X) \leq 4$, which admits a maximal unipotent minimal semistable degeneration, then $\mathrm{Sk}(X) \simeq \mathbb{S}^{\dim X}$.

Character Variety

Conjecture (Simpson)

Assume $X^0 = M_B$ is the character variety, of local systems on a k-punctured Riemann surface with fixed conjugacy classes of the monodromies around the punctures which satisfy some generic condition. Then $\mathcal{D}(E)$ is homotopic to a sphere if (X, E) is an snc compactificaiton of X^0.

- In many (all?) case X^0 can be compactified as a log CY pair (X, E), so we can ask whether $\mathcal{D}(E)$ is indeed a PL-sphere.
- (Simpson) The conjecture is true when the rank of the local system is 2 and the curve is $\mathbb{P}^1 \setminus \{p_1, ..., p_k\}$ $(k \geq 4)$.

Thank you very much!