Recent progress in the geometric Bogomolov conjecture

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Plan

§ 1 Introduction and the main result

Notation

Use the following notation in this talk.

k : algebraically closed field ("constant field").

$\mathfrak{B}$: normal projective variety over k.
Fix an ample line bundle $\mathcal{H}$ on $\mathfrak{B}$ when $\dim(\mathfrak{B}) \geq 2$.

K : the function field of $\mathfrak{B}$, or sometimes a finite number field

Then we have the notion of height over K.

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This $\widehat{h}_L$ is called the *canonical height* associated to L.

Properties of canonical heights

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Proposition

- *(Compatibility with tensor products) For any even* L_1, L_2, *we have* $\widehat{h}_{L_1 \otimes L_2} = \widehat{h}_{L_1} + \widehat{h}_{L_2}$.
- *(Compatibility with homomorphism) If* $f : B \to A$ *a homomorphism of abelian varieties and* L *is an even line bundle on* A, *then* $f^* \widehat{h}_L = \widehat{h}_{f^*(L)}$. *(Remark that the pullback of an even line bundle is again even.)*
- *(Positivity) If* L *is ample as well as even, then* $\widehat{h}_L \geq 0$.

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- *(Positivity) If L is ample as well as even, then $\widehat{h}_L \geq 0$.*

Remark

Since $\widehat{h}_L$ is a quadratic form, for $x \in A(\overline{K})_{tor}$, we have $\widehat{h}_L(x) = 0$.

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L : even ample line bundle on J
$\widehat{h}_L : J\left(\overline{K}\right) \to \mathbb{R}$: canonical height (remark $\widehat{h}_L \geq 0$)
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Conjecture (Bogomolov conjecture for curves)

When K is a function field, assume that C is non-isotrivial(, that is, there does not exists a curve C_0 over k s.t. $C \cong C_0 \otimes_k \overline{K}$). Then there exists an $\epsilon > 0$ such that $C(\epsilon; L)$ is finite.

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Assume that K is a number field. Then the Bogomolov conjecture holds, i.e, $\#C(\epsilon; L) < \infty$ for some $\epsilon > 0$.

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The conjecture over function fields is still open, though.

Theorem (Cinkir (2011))

Let K be the function field of $\mathfrak{B}$. Assume that $\dim(\mathfrak{B}) = 1$ (, so that K is a function field of tr.deg. 1,) and assume that $\mathrm{char}(k) = 0$. Then the Bogomolov conjecture holds, that is, if C is non-isotrivial, then $\#C(\epsilon; L) < \infty$ for some $\epsilon > 0$.

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Theorem (Y-, preprint (2015))

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Remark

To prove the above theorem, we need a generalized version of Bogomolov conjecture, called the geometric Bogomolov conjecture for abelian varieties (explained later).

§ 2 Zhang's theorem

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Suppose that X is a torsion subvariety(, that is, there exist an abelian subvariety G and a torsion point $\tau \in A(\overline{K})_{tor}$ such that $X = G + \tau$). Then for any $\epsilon > 0$, $X(\epsilon; L)$ is dense in X;

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Assume that K is a number field. Let L be ample and even. Then, if $X(\epsilon; L)$ is dense in X for any $\epsilon > 0$, then X is a torsion subvariety.

Note that this theorem generalizes Ullmo's.

Height 0 points

Assume that K is a number field. Let L be ample and even. Then it is classically known that for any $x \in A(\overline{K})$, if $\widehat{h}_L(x) = 0$, then x is a torsion point. Zhang's theorem claims the same thing holds for positive dimensional suvbvarieties.

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When K is a function field, A has height 0 points other than torsion points in general. This indicates that we cannot expect the same statement as Zhang's theorem over function fields in general.

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Let X be a closed subvariety of A. We say X *has dense small points* if $X(\epsilon; L)$ is dense in X for any $\epsilon > 0$, where L is even ample.

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§ 3 Geometric Bogomolov conjecture

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Then, in general, there is a closed subvariety X of A which is not a torsion subvariety but has dense small points. Such a phenomenon caused by contribution of the constant part of A.

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Lemma

Let $\phi : B \to A$ be a homomorphism. Then $\phi(Y)$ has dense small points.

To control such constant contribution, we use "trace".

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A pair $\left(\widetilde{A}^{\overline{K}/k}, \mathrm{Tr}_A^{\overline{K}}\right)$ of an abelian variety $\widetilde{A}^{\overline{K}/k}$ over k and a homomorphism $\mathrm{Tr}_A^{\overline{K}} : \widetilde{A}^{\overline{K}/k} \otimes_k \overline{K} \to A$ is called a $\overline{K}/k$-*trace* of A if it has the following universal property:

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It is known that there exists a $\overline{K}/k$-trace for any A, and it is unique up to a canonical isomorphism.

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Proposition (cf. Lang's book)

For $x \in A\left(\overline{K}\right)$, x has height 0 if and only if

$$x \in \mathrm{Tr}_A^{\overline{K}}\left(\widetilde{A}^{\overline{K}/k}(k)\right) + A\left(\overline{K}\right)_{tor}.$$

Special subvarieties

Definition

We say that X is *special* if there exist a closed subvarieties $\widetilde{Z} \subset \widetilde{A}^{\overline{K}/k}$, an abelian subvariety $G \subset A$ and a torsion point $\tau \in A\left(\overline{K}\right)_{tor}$ such that

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Remark

- A torsion subvariety is a special subvariety. If the $\overline{K}/k$-trace is trivial, then a special subvariety is a torsion subvariety.
- A special subvariety has dense small points. In fact, with the notation above, $\mathrm{Tr}_A^{\overline{K}/k}\left(\widetilde{Z}(k)\right) + G\left(\overline{K}\right)_{tor} + \tau$ is dense in X, and any point of this set has height 0.

Geometric Bogomolov conjecture

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Recall the BC for curves over function fields: Suppose that C is non-isotrivial; then C should not have dense small points (as a subvariety of its Jacobian J). Note that the BC for curves over function fields is a part of GBC for abelian varieties, because non-isotrivial curve C of genus $g \geq 2$ cannot be special in J.

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Remark

Since $\mathfrak{m} = 0$ in Gubler's setting, Theorem 3.7 generalizes Gubler's theorem of totally degenerate setting.

§ 4 Structure of the proof of the main result

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From "codim 1 case" to "dim 1 case" (by contradiction): Suppose that there exists X s.t. $\dim(X) = 1$, X non-special, and with dense small points. Then (after some reduction steps), $Y := X + \cdots + X \subset A$ can be a non-special subvariety of codimension 1 of A with dense small points, which contradicts

Three steps of the proof

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The above theorem is one of the key steps, and its proof uses non-archimedean geometry quite essentially.

§ 5 Non-archimedean geometric part

Goal in the remaining

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To explain an idea of the proof of the above theorem, we assume for simplicity that:

- A is simple;
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Let X be a closed subvariety of A. Suppose that X has dense small points. Then we want to prove that X consists of a torsion point.

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X : closed subvariety of A
$d := \dim(X)$

$c_1(\overline{L}|_X)^d$: Chambert-Loir measure on X^{an} associated to $\overline{L}|_X$,
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Remark

One sees that the notion of strict support does not depend on L.

Semistable alteration and the measures I

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$\mathrm{str}(\tilde{\mathscr{X}}')$: set of strata of $\tilde{\mathscr{X}}'$
Δ_S : canonical simplex associated to $S \in \mathrm{str}(\tilde{\mathscr{X}}')$
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Proposition (Gubler (2010))

$c_1(f^*\overline{L})^d = \sum_{S \in \mathrm{str}(\tilde{\mathscr{X}}')} r_S \delta_{\Delta_S} \quad (r_S \geq 0).$

Remark

"$r_S > 0$" does not depend on L.

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Proposition (Gubler (2010))

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Lemma (Y-, crucial lemma)

Let σ *be a strict support of a canonical measure. Then for any* Δ_S *with* $\sigma \subset f(\Delta_S)$ *and* $\dim(f(\Delta_S)) = \dim(\sigma)$*, we have* $\dim(\sigma) = \dim(\Delta_S)$ *(and in fact,* $r_S > 0$ *furthermore).*

Proof I

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Suppose that X is not a torsion subvariety but has dense small points (arguing by contradiction). For $N \geq 2$, let $A^N \to A^{N-1}$ be the homomorphism given by $(x_1, \dots, x_N) \mapsto (x_1 - x_2, \dots, x_{N-1} - x_N)$, and let Y be the image of $X^N \subset A^N$. Since X has trivial stabilizer, there exists an $N > 0$ s.t. the restriction morphism $\beta : X^N \to Y$ is a generically finite surjective morphism.

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Let $\mu_{X_v^N}$ and μ_{Y_v} be canonical measures, where "$_v$" indicates the associated analytic spaces at v. Then we can show $\beta_* \mu_{X_v^N} = \mu_{Y_v}$ by the theorem of equidistribution of small points (Szpiro–Ullmo–Zhang, Gubler).

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- Since $\beta(\sigma)$ is a strict support, the crucial lemma shows us $\dim \Delta_S = \dim(\beta(\sigma))$, and hence $\dim \beta(\sigma) = \dim \sigma$; contradiction.